Optimal Execution under Liquidity Uncertainty

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Abstract

We study an optimal execution strategy for purchasing a large block of shares over a fixed time horizon. The execution problem is subject to a general price impact that gradually dissipates due to market resilience. This resilience is modeled through a potentially arbitrary limit-order book shape. To account for liquidity dynamics, we introduce a stochastic *volume effect* governing the recovery of the *deviation process*, which represents the difference between the impacted and unaffected price. Additionally, we incorporate stochastic liquidity variations through a regime-switching Markov chain to capture abrupt shifts in market conditions. We study this singular control problem, where the trader optimally determines the timing and rate of purchases to minimize execution costs. The associated value function to this optimization problem is shown to satisfy a system of variational Hamilton–Jacobi–Bellman inequalities. Moreover, we establish that it is the unique viscosity solution to this HJB system and study the analytical properties of the free boundary separating the execution and continuation regions. To illustrate our results, we present numerical examples under different limit-order book configurations, highlighting the interplay between price impact, resilience dynamics, and stochastic liquidity regimes in shaping the optimal execution strategy.

Keywords: Optimal Execution, Singular Control, Viscosity Solutions, Free Boundary Problem, Regime Switching.

Mathematical subject classifications: 93E20, 49L25, 91B70.

1 Introduction

The problem of optimal execution in financial markets has been extensively studied over the past decades, with early research focusing on deterministic and linear price impact models. Classical approaches assumed that market participants could execute large trades with predictable and stable costs, often modeling price impact as a linear function of trading volume as in Bertsimas and Lo [9] and Almgren [4]. These models provided foundational insights into the trade-off between execution cost and market risk, establishing a framework in which traders optimize their execution trajectories over a fixed time horizon. However, empirical studies such as Bouchaud, Farmer, and Lillo [11] and Zhou [44], have shown that the assumption of a simple linear price impact does not accurately reflect real-world market dynamics. Market impact is often nonlinear, transient, and influenced by complex interactions within the limit order book (LOB).

As a result, recent research has moved beyond simplistic assumptions, incorporating more advanced models that better capture this non-linearity. Obizhaeva and Wang [34] introduced a

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framework where market impact depends explicitly on the LOB shape, showing that execution strategies should adapt to liquidity fluctuations. Alfonsi, Fruth, and Schied [3] extended this idea by considering a continuous LOB model where price impact depends on both the execution rate and liquidity replenishment. These advancements laid the foundation for further research on execution strategies, notably Alfonsi and Blanc [2], Ekren and Muhle-Karbe [24], and Cayé, Herdegen, and Muhle-Karbe [15]. A significant contribution in this direction was made by Predoiu, Shaikhet, and Shreve [37], who analyzed optimal execution in a one-sided limit order book with nonlinear price impact and resilience. Their model, which accommodates an arbitrary order book shape, derives the optimal strategy as a sequence of lump-sum trades interspersed with continuous trading at a rate matching order book resiliency. These results formalized the interaction between execution strategies and market recovery, providing a framework for modeling price impact in order-driven markets. However, the assumption of deterministic resilience limits its applicability, as stochastic fluctuations in liquidity have been well documented; see for instance in Taranto et al. [43].

Building on this, Becherer, Bilarev, and Frentrup [8] studied a two-dimensional singular control problem of finite fuel type over an infinite time horizon, motivated by the optimal liquidation of an asset position in a financial market with multiplicative and transient price impact. This framework incorporates stochastic liquidity, where the *volume effect* process is driven by its own random noise, and they derive an explicit closed-form solution under the assumption of multiplicative price impact. Fruth, Schoneborn, and Urusov [27] introduced a stochastic order book depth, addressing the limitations of models that assume constant or deterministically varying liquidity. They establish the existence and uniqueness of optimal execution strategies when order book depth follows a general diffusion process. Unlike prior work assuming deterministic liquidity variations, this framework captures stochastic liquidity fluctuations. More recently, Ackermann, Kruse, and Urusov [1] explored optimal execution in a market where order book depth and resilience evolve stochastically. The authors formulated a discrete-time trading model and derive a recursive equation for the minimal expected execution cost. The authors analyzed conditions under which traders should execute immediately or gradually. Horst and Xia [29] studied the optimal liquidation of multi-asset portfolios in markets where resilience is stochastic. The authors developed a framework that accounts for both immediate and persistent price impacts, formulating the problem using backward stochastic Riccati differential equations (BSRDEs) to characterize the value function. To handle the singular terminal conditions in these equations, they introduced an approximation method via penalization, solving a sequence of unconstrained problems that enforce full liquidation. Their results provide explicit optimal trading strategies. Finally, Fouque, Jaimungal, and Saporito [26] examined optimal trading in markets where price impact is stochastic and often exhibits fast mean reversion. Using singular perturbation methods, the authors developed approximations to the resulting optimal control problem and established their accuracy by constructing sub- and super-solutions. The analysis highlights how stochastic trading frictions influence execution strategies. Muhle-Karbe, Wang, and Webster [33] demonstrate that in concave price impact models a tractable linear approximation emerges in the high-frequency limit. The authors show that a stochastic liquidity parameter effectively captures the model's nonlinearity, yielding a diffusion limit that preserves key features and performs well on limit order book data.

In this work, we introduce a stochastic volume effect governed by a jump diffusion process, allowing the model to capture both continuous liquidity variation and sudden market shocks. The price impact function is allowed to be discontinuous, accommodating abrupt changes in market conditions. Instead of specifying a particular model for the fundamental price, we require only a martingale property, making the framework applicable under general market dynamics. Liquidity regimes are incorporated through a finite-state Markov chain, reflecting structural shifts in market depth. We formulate the resulting optimization as a singular stochastic control problem. Unlike most singular stochastic control problems, which are typically posed in infinite horizon settings (see, e.g., Shreve and Soner [42], Bank and El Karoui [6]), our problem is inherently finite-horizon due to the nature of execution tasks. In the execution literature, problems are often formulated either in discrete time or as impulse controls over a finite horizon, highlighting the distinctive

structure of our continuous-time formulation. Moreover, our model avoids explicit cost functions often used in classical formulations, as our setting focuses on liquidity-induced frictions rather than inventory or running penalties. To the best of our knowledge, the closest related work is that of Predoiu, Shaikhet, and Shreve [37], which addresses a similar execution setting but in a static framework, whereas our approach is fully dynamic.

We incorporate different market regimes to model liquidity variations more realistically. Empirical evidence, such as Chevalier, Hafsi, and Ly Vath [17], shows that market liquidity distribution varies significantly throughout the trading day. Studies like Bayraktar and Ludkovski [7], Colaneri et al. [19], Chung Siu et al. [18], Dammann and Ferrari [22], Cartea and Sánchez-Betancourt [14] and Chevalier, Hafsi, and Ly Vath [16] have addressed this gap by modeling liquidity as a stochastic process. To capture this, we model liquidity regimes as a finite state Markov chain. In our framework, liquidity regime variations affect only the shape of the limit order book, while the underlying *volume effect* dynamics remain unchanged. This ensures that price evolution is not exogenously driven by regime shifts but instead results from structural changes in supply and demand at different liquidity levels. By allowing the LOB's form to vary across regimes, we account for regime-dependent price impact, where trades in low-liquidity states induce greater price responses than in high-liquidity conditions. This approach captures transient liquidity shocks, aligns with empirical evidence, and provides a more precise characterization of execution risk.

Finally, we address an optimal execution problem formulated as a singular control problem. Our approach is guided by techniques from Ma et al. [31], Liu [30], and Shao and Tian [41], which connect regime-switching models with the first component of the Hamilton-Jacobi-Bellman Quasi-Variational Inequalities (HJBQVI). These works reformulate the execution problem as a stochastic control problem for a Markov jump linear system, deriving the value function and optimal feedback strategy through coupled differential Riccati equations. We also formulate our study as a free boundary problem in the spirit of Caffarelli [13], Pham [35] and Figalli and Serra [25], focusing on obstacle-type formulations and regularity properties. Due to the multi-dimensionality of the state space, classical verification techniques are not applicable, as the value function and optimal control cannot be characterized in closed form. To address this, we adopt a viscosity solution framework and characterize our value function as the unique solution to the HJB system. This allows us to describe the structure of the continuation and intervention regions, and to show the existence of a connected free boundary separating them. Due to the lack of sufficient regularity properties inherent in our setting, we rely on approximation methods to construct the solution. We illustrate our findings with numerical examples across various limit order book configurations. The viscosity solution is approximated using an implicit-explicit finite difference scheme for the PIDE, following Cont and Voltchkova [20]. The simulations show that higher market activity reduces market impact. In particular, increased volatility and resilience in the volume effect lead to greater liquidity and lower exercise boundaries, while the opposite holds in less active markets. We also examine how different limit order book shapes influence the free boundary.

The article is organized as follows. Section 2 introduces the model describing the price and liquidity dynamics of a single asset, with a LOB featuring stochastic resilience and accounting for the stochastic market liquidity. This section also formulates the singular control problem associated with the execution strategy. Section 3 examines the analytical properties of the value function and establishes its continuity over the domain. In Section 4, we demonstrate that the value function uniquely solves the system of variational inequalities as a viscosity solution and prove the connectedness of the free boundary. We present the numerical scheme to approximate the viscosity solution in Section 5. Finally, Section 6 presents numerical illustrations, analyzing the influence of state variables and price impact on the value function and the free boundary.

Notation 1.1. *For $x \in \mathbb{R}^k$, where k is determined by the context, $\|x\|$ denotes its Euclidean norm, and $B_r(x)$ represents the open ball centered at x with radius $r > 0$. The scalar product is denoted by $\langle \cdot, \cdot \rangle$, and for a vector $x \in \mathbb{R}^k$, its transpose is denoted by x^T. For a set $A \subset \mathbb{R}^k$, ∂A denotes its boundary, while $\mathrm{int}(A)$ or equivalently $\mathring{A}$ denote its interior. The symbol $\otimes$ represents*

the Kronecker product of two matrices. For a function $\varphi : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}$, the gradient and Hessian matrix are denoted by $D\varphi$ and $D^2\varphi$, respectively, whenever they are well-defined.

2 Optimal Execution Problem

Consider a complete filtered probability space $(\Omega, \mathbb{F}, \mathcal{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, where $\{\mathcal{F}_t\}_{t\geq 0}$ is a right-continuous filtration. We assume that the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ supports a standard $\mathbb{P}$-Brownian motion W, a continuous Markov chain I, and a random Poisson measure M on $\mathbb{R}_+ \times \mathbb{R}_+$ with compensator $\lambda_t \nu(\mathrm{d}z)\mathrm{d}t$, where $\lambda : [0, T] \to [0, \bar{\lambda}]$, and ν is a measure on $\mathbb{R}$. Throughout this paper, unless explicitly stated otherwise, any equality between random variables is understood to hold $\mathbb{P}$-almost surely when it is not explicitly stated.

2.1 Model Setup

We consider $\{I_t\}_{t\geq 0}$ to be a continuous-time Markov chain with values in a finite state space $\mathbb{I}_m := \{1, \ldots, m\} \subset \mathbb{N}$ and with càdlàg sample paths. We denote $\mathcal{F}^I = \{\mathcal{F}_t^I\}_{t\geq 0}$ the natural filtration generated by I, augmented by the $\mathbb{P}$-null sets. The process I represents the liquidity state within the limit order book. Its infinitesimal generator $Q(t) = (Q_{ij}(t))_{1\leq i,j \leq m}$ captures the intensities of transitioning between different liquidity states. The off-diagonal elements $Q_{ij}(t)$ for $i \neq j$ correspond to the continuous and bounded instantaneous rate of transition from state i to state j. We assume that I is stable and conservative, i.e., for all $(j, k) \in \mathbb{I}_m^2$ and $t \geq 0$,

$$Q_{jj}(t) = -\sum_{k \neq j} Q_{jk}(t), \quad \text{and} \quad Q_{jj}(t) < +\infty. \tag{1}$$

Volume Effect. Let $i \in \mathbb{I}_m$ be a liquidity regime. We define $A := (A_t)_{t\geq 0}$ as the reference price of the assets, which we assume to be a continuous $(\mathbb{P}, \mathcal{F})$-martingale. In our model, we assume that, in the absence of trading, the number of available shares at time t in the price interval $[A_t, A_t + x[$ is $F_{I_t}(x)$. Here, F_i is a non-decreasing and left-continuous function associated to an infinite measure $\mu_i : \mathbb{R}_+ \to \mathbb{R}_+$ in the following way:

$$F_i(x) := \mu_i([0, x[), \quad \forall x \in \mathbb{R}_+.$$

The shape of the *shadow limit order book* (LOB) dynamically evolves as the liquidity regime changes. This allows the model to adapt to varying market conditions.

Remark 2.1. *This model is general and accommodates various limit order book structures and market impacts. Examples include block, modified block, and discrete order books (see Predoiu, Shaikhet, and Shreve [37]). It also supports other types, such as power-law order books (see Brokmann et al. [12] and Avellaneda and Stoikov [5]).*

Let Y be an $\mathcal{F}$-adapted nonnegative process representing the *volume effect* process, such that

$$\begin{cases} \mathrm{d}Y_u = -h(Y_{u-})\mathrm{d}u + \sigma(Y_{u-})\mathrm{d}W_u + \int_{\mathbb{R}} q(Y_{u-}, z)(M(\mathrm{d}u, \mathrm{d}z) - \lambda_u \nu(\mathrm{d}z)\mathrm{d}u), \\ Y_{t-} = y, \end{cases} \tag{2}$$

with $t \geq 0$, $u \in [t, T]$, $y \geq 0$, $h : \mathbb{R}_+ \to \mathbb{R}$, $\sigma : \mathbb{R}_+ \to \mathbb{R}$ and $q : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$. We denote

$$\check{Y}_{u-} := Y_{u-} + \Delta_M Y_u,$$

where $\Delta_M Y_u$ represents the jump contribution of the measure M to Y at time u.

Assumptions 2.1. *We assume that the processes I and M do not have mutual covariation.*

Remark 2.2. *Assumption 2.1 simplifies the analysis without minimal loss of generality, as any dependence can typically be absorbed into the dynamics of M or handled via a change of measure.*

The ask price is no longer the reference price but is given by $P := A + D$. Here, D defines the price deviation process, such that $D_u := \psi_{I_u}(Y_u)$ for all $u \geq 0$. The function ψ_i is left-continuous and verifies

$$\psi_i(y) := \sup\{a \geq 0 : F_i(a) < y\}, \quad \forall y \geq 0, \tag{3}$$

with $\psi_i(0) := 0$. It follows from the monotonicity and the left-continuity of F_i that ψ_i is non-decreasing on $\mathbb{R}_+$ and that $F_i(\psi_i(y)) = y$, for all $y \in \mathbb{R}_+$.
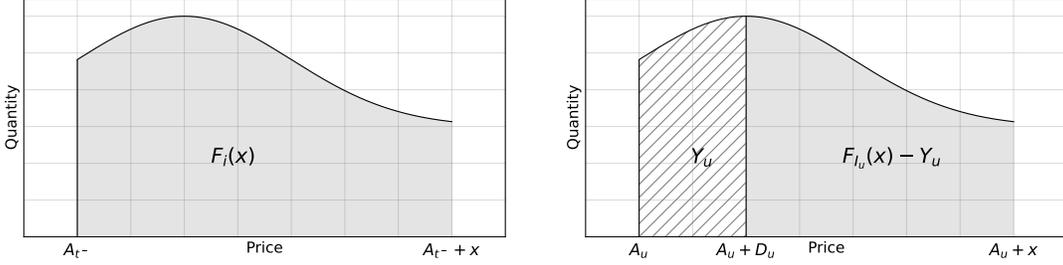

Figure 1: Representation of the limit order book at times t^- and u. The left panel illustrates the initial state with $Y_{t^-} = y = 0$, while the right panel shows the LOB after transactions occurring between t^- and u. The shaded area Y_u indicates the volume of consumed shares.

The volume consumption affecting the LOB also accounts for the influence of other market participants, modeled through the Brownian motion W and the compensated martingale $\overline{M}$, with $\overline{M}(\mathrm{d}u, \mathrm{d}x) := M(\mathrm{d}u, \mathrm{d}z) - \lambda_u \nu(\mathrm{d}z)\mathrm{d}u$. The jumps here capture the impact of block trades and large orders placed by external market participants.

We list the following assumptions to support our study of the control problem later.

Assumptions 2.2. *Assume that:*

(A1) There exists $b > 0$, $\beta > 0$ and $a > 0$ such that

$$F_i(x) \geq bx^{\beta}, \quad \forall(x, i) \in [a, +\infty[\times\mathbb{I}_m.$$

(A2) The measure $\nu(\mathrm{d}z)$ satisfies

$$\int_{\mathbb{R}}(1 + z^2)\nu(\mathrm{d}z) < +\infty.$$

(A3) There exists a constant $C > 0$ such that, for all $y \geq 0$,

$$|h(y)| + |\sigma(y)| + \int_{\mathbb{R}}|q(y, z)|\nu(\mathrm{d}z) \leq C(1 + |y|).$$

(A4) There exists a constant $L > 0$ such that, for all $y, y' \geq 0$,

$$|h(y) - h(y')| + |\sigma(y) - \sigma(y')| + \int_{\mathbb{R}}|q(y, z) - q(y', z)|\nu(\mathrm{d}z) \leq L|y - y'|.$$

Assumption (A1) prevents the placement of buy orders with finite size at infinitely high prices. Moreover, it provides an upper bound for the resilience function ψ_i, which is crucial for establishing the joint continuity of the value function introduced in the following sections. It also plays a key role in proving the Comparison Principle (see Theorem 4.3) for the viscosity solutions of the value function. Assumptions (A2), (A3), and (A4) guarantee the existence and uniqueness of a strong solution to the SDE (2) via Proposition 2.1, as well as a comparison property for the *volume effect* process through Lemma 1. These assumptions hold for the rest of this study.

2.2 Problem Formulation

Our goal is to address an optimal purchase problem for a single asset over a finite time horizon T. As this study is conducted in a continuous trading context rather than high-frequency trading, we consider singular controls as admissible strategies, where jumps correspond to block trades.

A financial agent wants to buy $\overline{X}$ shares of an illiquid asset over the time interval $[0, T]$. The agent's holdings X form an $\mathcal{F}$-progressively measurable process.

Definition 2.1 (Admissible strategies). *Define $\mathcal{A}_t(x)$ as the set of admissible controls for an agent with a position $x \in \mathbb{R}_+$ at time $t \in [0, T]$. An admissible purchase strategy consists of a non-decreasing $\mathcal{F}$-adapted right-continuous process $X = (X_u)_{t \leq u \leq T}$ such that $X_{t^-} = x$ and $X_T = \overline{X}$. We denote by $\Delta X_u = X_u - X_{u^-}$ the jump at time u. We also represent X^c as the continuous component of X.*

We define the controlled dynamics of the *volume effect* process $Y^{t,y,X}$ as

$$
\begin{cases}
\mathrm{d}Y_u^{t,y,X} = \mathrm{d}X_u - h(Y_{u^-}^{t,y,X})\mathrm{d}u + \sigma(Y_{u^-}^{t,y,X})\mathrm{d}W_u + \int_{\mathbb{R}} q(Y_{u^-}^{t,y,X}, z)(M(\mathrm{d}u, \mathrm{d}z) - \lambda_u \nu(\mathrm{d}z)\mathrm{d}u), \\
Y_{t^-}^{t,y,X} = y,
\end{cases}
\tag{4}
$$

with $u \in [t, T]$ and $y \geq 0$. Using classical theory (see Protter [38]), we can directly conclude that the given system of stochastic differential equations has a unique strong solution.

Proposition 2.1. *For any $\mathcal{F}_t$-measurable random variable ξ valued in $\mathbb{R}_+$ such that $\mathbb{E}(|\xi|^p) < +\infty$, for some $p > 1$, the SDE (4) admits, for all $t \in [0, T]$, a unique strong solution $Y^{t,\xi,X}$, with $Y_t^{t,\xi,X} = \xi$. Moreover,*

$$
\mathbb{E}\Big[\sup_{0 \leq u \leq T} |Y_u^{t,y,X}|^p\Big] \leq C_T\big(1 + \mathbb{E}(|\xi|^p)\big),
$$

where the constant $C_T \in \mathbb{R}_+$ only depends on T, $\overline{X}$ and the Lipschitz coefficients of h, σ and q.

We state a comparison result for the sample paths of Y, which will be used in proving the monotonicity of v and later in the proof of continuity.

Lemma 1 (Comparison property). *Let $t \in [0, T]$, $x \in [0, \overline{X}]$ and $X \in \mathcal{A}_t(x)$. For any two controlled solutions $Y^{t,Y_0^1,X}$ and $Y^{t,Y_0^2,X}$ of the SDE (4) satisfying $Y_0^1 \leq Y_0^2$, it holds that*

$$
\mathbb{P}\big(Y_u^{t,Y_0^1,X} \leq Y_u^{t,Y_0^2,X}, \quad \forall u \in [t, T]\big) = 1.
$$

Proof. Refer to Theorem 2.3 in Dawson and Li [23]. $\qquad\square$

Note that the *volume effect* process $Y^{t,y,X}$ incorporates the impact of the investor's trading X within its dynamics. Equation (4) specifies how the order book is affected by the transactions of the trader. If a large transaction happens at time t with the investor buying x shares, then $Y_t^{t,y,X} = \check{Y}_t^{t,y,X} + x$. The ask price jumps from $A_{t^-} + \psi_{I_{t^-}}(Y_{t^-}^{t,y,X})$ to $A_t + \psi_{I_t}(\check{Y}_t^{t,y,X} + x)$ after the transaction. Based on the limit order book structure described in Section 2.1, the purchase cost for a large trade of size y, in excess of A_t, at time t is expressed, for every regime $i \in \mathbb{I}_m$, as

$$
\begin{aligned}
\Phi_i(y) &:= \int_0^{\psi_i(y)} \xi \mathrm{d}F_i(\xi) \\
&= \int_0^y \psi_i(\zeta)\, \mathrm{d}\zeta, \quad \forall y \geq 0.
\end{aligned}
\tag{5}
$$

In other words, the cost of placing a buy order of size y at time t at price P_t is given by

$$
\pi_t^i(y) = \int_0^y \big[P_t + \psi_i(\zeta)\big]\mathrm{d}\zeta = \underbrace{P_t y}_{\text{Cost at the current price}} + \underbrace{\Phi_i(y)}_{\text{Impact cost}}.
$$

We assume that the Markov chain I starts in state i at time t. Since X is nonnegative and bounded by $\overline{X}$, with $X_{t^-} = x$, we express the cost of the purchase strategy X over $[t, T]$, under suitable conditions on the price A, as described in Predoiu, Shaikhet, and Shreve [37]. In this case, the cost is equal to

$$\int_t^T (A_u + \check{D}_{u^-}) \mathrm{d}X_u^c + \sum_{t \leq u \leq T} A_u \Delta X_u + \Phi_{I_u}(Y_u^{t,y,X}) - \Phi_{I_{u^-}}(\check{Y}_{u^-}^{t,y,X})$$

$$= \int_t^T \psi_{I_{u^-}}(\check{Y}_{u^-}^{t,y,X}) \mathrm{d}X_u^c + A_t(\overline{X} - x) + \sum_{t \leq u \leq T} \Phi_{I_u}(Y_u^{t,y,X}) - \Phi_{I_{u^-}}(\check{Y}_{u^-}^{t,y,X}),$$

where $\check{D}_u := \psi_{I_u}(\check{Y}_u^{t,y,X})$ and $u \in [t, T]$. Consequently, assuming the common condition that the ask price A is a martingale, minimizing slippage is equivalent to minimizing the total cost of purchases. This leads to the following optimization problem, where we aim to minimize the expected cost in excess of $A_t(\overline{X} - x)$ over the set of admissible strategies, defined by the value function $v := (v_i)_{i \in \mathbb{I}_m}$, such that

$$v_i(t,x,y) := \inf_{X \in \mathcal{A}_t(x)} \mathbb{E}\Big[\int_t^T \psi_{I_u}(\check{Y}_{u^-}^{t,y,X}) \mathrm{d}X_u^c + \sum_{t \leq u \leq T} \Phi_{I_u}(Y_u^{t,y,X}) - \Phi_{I_{u^-}}(\check{Y}_{u^-}^{t,y,X}) \Big], \qquad (6)$$

with $(t,x,y) \in \mathcal{S} := [0,T[\times [0, \overline{X}[\times \mathbb{R}_+^*$ and $i \in \mathbb{I}_m$. We denote $\bar{\mathcal{S}}$ as the closure of $\mathcal{S}$.
The boundary conditions are defined by the immediate purchase of the remaining shares as

$$v_i(T,x,y) = \Phi_i(y + \overline{X} - x) - \Phi_i(y),$$
$$v_i(t,\overline{X},y) = 0. \qquad (7)$$

Note that v_i satisfies the following growth condition (see Proposition 3.1)

$$0 \leq v_i(t,x,y) \leq \Phi_i(y + \overline{X} - x) - \Phi_i(y). \qquad (8)$$

2.3 Examples

We begin by presenting examples that inspired our study. For clarity, these illustrations focus on the single-regime case ($m = 1$), although we consider the general setting throughout the paper.

Let $(t,x,y) \in \bar{\mathcal{S}}$, and $X \in \mathcal{A}_t(x)$. Assume that the *volume effect* is deterministic, meaning $\sigma \equiv q \equiv 0$. Further, suppose that process $Y^{t,y,X}$ has finite variation and satisfies $Y_{t^-}^{t,y,X} = y$. Under these conditions, the setup corresponds to Model 1 in Alfonsi, Fruth, and Schied [3] within a discrete-time framework and, more specifically, to the one detailed in Predoiu, Shaikhet, and Shreve [37]. Applying Itô's formula, we obtain

$$\Phi(Y_T^{t,y,X}) - \Phi(y) = \int_t^T \psi(Y_u^{t,y,X})\big(\mathrm{d}X_u^c - h(Y_u^{t,y,X})\mathrm{d}u\big) + \sum_{t \leq u \leq T} \Phi(Y_u^{t,y,X}) - \Phi(Y_{u^-}^{t,y,X}).$$

Hence,

$$v(t,x,y) = \inf_{X \in \mathcal{A}_t(x)} \mathbb{E}\Big[\Phi(Y_T^{t,y,X}) + \int_t^T g(h(Y_u^{t,y,X}))\mathrm{d}u \Big] - \Phi(y),$$

where

$$g(y) = y\psi(h^{-1}(y)).$$

If one further assumes that g is a convex function, Jensen's inequality implies that

$$\int_t^T g(h(Y_u^{t,y,X}))\mathrm{d}u \geq (T-t)g\Big(\frac{1}{T-t} \int_t^T h(Y_u^{t,y,X})\,\mathrm{d}u \Big).$$

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Knowing that $Y_T^{t,y,X} = y + \overline{X} - x - \int_t^T h(Y_u^{t,y,X}) \, du$, we get

$$v(t,x,y) \geq \inf_{X \in \mathcal{A}_t(x)} \mathbb{E}\left[G_{t,x,y}(Y_T^{t,y,X}) \right] - \Phi(y),$$

where $G_{t,x,y}(e) = \Phi(e) + (T-t)g\left(\frac{\overline{X}-x+y-e}{T-t}\right)$. We define a type A strategy as a strategy that has an initial jump at time t, satisfying $dX_u = dX_u^c = h(Y_u)du$ on $]t,T[$, with the agent unwinding the remaining position at time T. Assume that $e_{t,x,y}^*$ minimizes $G_{t,x,y}$, and

$$e_{t,x,y}^* \in \{Y_T^{t,y,X} : X \in \mathcal{A}_t(x) \text{ is of type } A\}.$$

Then,

$$v(t,x,y) = G(e_{t,x,y}^*) - \Phi(y). \tag{9}$$

Theorem 4.2 of Predoiu, Shaikhet, and Shreve [37] establishes that the type A strategy associated to $e_{t,x,y}^*$ is the unique solution to the optimization purchase problem when g is strictly convex. Furthermore, the authors demonstrate that a solution to the optimal execution problem exists even if the convexity of g no longer holds (see Theorem 4.5). Under this alternative situation, the trader initially purchases a lump sum of shares, $X_0 = y$, at time 0. Subsequently, shares are bought at a constant rate, $dX_t = h(y) \, dt$, over the interval $t \in [0, t_0[$, maintaining $Y_t = y$. At time t_0, another lump sum purchase is made, after which the trader continues to buy shares at a constant rate, $dX_t = h(Y_{t_0}) \, dt$, during $t \in [t_0, T[$, keeping $Y_t = Y_{t_0}$. Finally, the remaining shares are purchased at time T. For example, consider a block-shaped order book as described in Obizhaeva and Wang [34], with a density $q_0 = 1$ and a resilience rate $\rho = 1$, resulting in $F(x) = \psi(x) = h(x) = x$, for all $x \geq 0$. In this case, $\Phi(x) = \frac{x^2}{2}$ and $g(x) = x^2$, for all $x \geq 0$. Additionally, if $X \in \mathcal{A}_t(x)$ is of type A, the following holds

$$e_{t,x,y}^* = \frac{2(\overline{X} - x + y)}{T - t + 2}.$$

In the framework proposed by Predoiu, Shaikhet, and Shreve [37], the volume effect is assumed to be non-dynamic, effectively treating it as deterministic throughout the execution horizon. As a result, the authors implicitly assume that the initial state lies within the execution region when starting from $x = 0$ and $y = 0$. However, this assumption does not necessarily hold when the volume effect is dynamic. In general, $e_{t,x,y}^*$ is attainable by a type A strategy if and only if

$$\overline{X} - x - (1 + (T - t))y \geq 0.$$

Furthermore, the value function in this region is equal to

$$v(t,x,y) = \frac{(\overline{X} - x + y)^2}{2 + T - t} - \frac{y^2}{2}.$$

The representation of the value function in the next section enables its characterization beyond this domain. Observe that this case falls within the framework of Alfonsi and Blanc [2] if we assume that the strategic trader's proportion of permanent price impact is identical to that of other traders and equal to zero. Suppose further that the measure m satisfies $m \equiv 0$, the jump size of Y satisfies $q \equiv 1$, and that the process

$$\int_0^t \sigma^2(Y_u) \, dW_u$$

is $\mathcal{F}$-martingale, where Y is the solution of the uncontrolled SDE (2). The argument in the proof of Theorem 2.1 of Alfonsi and Blanc [2] shows that there are no Price Manipulation Strategies (PMS) in this setting and that the control problem is equivalent to the execution problem of Obizhaeva and Wang [34]. This example motivates our study of the general case, in which the volume effect evolves

dynamically as a jump diffusion process. In this setting, we aim to characterize the properties of the value function within each region and along the free boundary.

Remark 2.3. *The framework introduced by Alfonsi and Blanc [2] was extended by Chevalier, Hafsi, and Ly Vath [16] to incorporate impulse controls. Although this extension enhances the modeling of price impact, the limit order book remains static, meaning that its shape does not respond to incoming orders, and exhibits a deterministic volume effect with fixed resilience. Additionally, the dynamics are solely driven by point processes representing order flow. As a result, while these models are well suited to high-frequency trading environments, they are less appropriate for longer time scales. This motivates the use of Brownian noise in (2) to capture stochastic resilience effects.*

3 Analytical Properties of the Value Function

In this section, we will denote, for any $(i, t, x, y) \in \mathbb{I}_m \times \bar{\mathcal{S}}$, and any admissible strategy $X \in \mathcal{A}_t(x)$,

$$
C_i(t, x, y, X) := \int_t^T \psi_{I_u}(\check{Y}_{u^-}^{t,y,X}) \mathrm{d}X_u^c + \sum_{t \le u \le T; \Delta X_u > 0} \Phi_{I_u}(Y_u^{t,y,X}) - \Phi_{I_{u^-}}(\check{Y}_{u^-}^{t,y,X}).
$$

We start by stating standard results on the finiteness and monotonicity of the value function.

Proposition 3.1. *The value function v described in (6) is finite.*

Proof. Let $(t, x, y) \in \bar{\mathcal{S}}$ be the state variable at time t and $i \in \mathbb{I}_m$. If the investor buys $\bar{X} - x$ immediately, the associated cost would be equal to $\Phi_i(y + \bar{X} - x) - \Phi_i(y)$. Therefore,

$$
0 \le v_i(t, x, y) \le \Phi_i(y + \bar{X} - x) - \Phi_i(y)
$$

on $\bar{\mathcal{S}}$. This concludes the proof. $\qquad\square$

Proposition 3.2 (Monotonicity). *For any $(i, t, x, y) \in \mathbb{I}_m \times \bar{\mathcal{S}}$, the following results hold:*

1. *$t_0 \mapsto v_i(t_0, x, y)$ is non-decreasing on $[0, T]$.*

2. *$x_0 \mapsto v_i(t, x_0, y)$ is non-increasing on $[0, \bar{X}]$.*

3. *$y_0 \mapsto v_i(t, x, y_0)$ is non-decreasing on $\mathbb{R}_+$.*

Proof. **1.** Let $i \in \mathbb{I}_m$, $x \in [0, \bar{X}]$ and $y \ge 0$. Suppose that $0 \le t \le t' \le T$ and consider $X' = (X'_u)_{t' \le u \le T} \in \mathcal{A}_{t'}(x)$ such that $X'_{t'^-} = x$ and $X'_T = \bar{X}$. Using the strong Markov property, we get that

$$
\mathbb{E}\big[C_i(t', x, y, X') | \mathcal{F}_{t'}\big] = \mathbb{E}\Big[\int_t^{T-t'+t} \psi_{I_u}(\check{Y}_{u^-}^{t,y,\widetilde{X}'}) \mathrm{d}\widetilde{X}_u^c + \sum_{\substack{t \le u \le T-t'+t \\ \Delta \widetilde{X}_u > 0}} \Phi_{I_u}(Y_u^{t,y,\widetilde{X}'}) - \Phi_{I_{u^-}}(\check{Y}_{u^-}^{t,y,\widetilde{X}'}) | \mathcal{F}_t\Big],
$$

with $\widetilde{X}' = (X'_{u-t+t'})_{t \le u \le T-t'+t}$. Define the strategy $X \in \mathcal{A}_t(x)$ as

$$
X_u = \begin{cases} \widetilde{X}_u, & \text{if } u \in [t, T - t' + t], \\ 0, & \text{else.} \end{cases}
$$

Note that $\mathbb{E}\big[C_i(t', x, y, X') | \mathcal{F}_{t'}\big] = \mathbb{E}\big[C_i(t, x, y, X) | \mathcal{F}_t\big]$. Since $X \in \mathcal{A}_t(x)$, then

$$
v_i(t, x, y) \le \mathbb{E}\big[C_i(t', x, y, X') | \mathcal{F}_{t'}\big].
$$

As X' is arbitrary, it follows that $v_i(t, x, y) \le v_i(t', x, y)$, completing the proof.
2. Let $i \in \mathbb{I}_m$, $t \in [0, T]$ and $y \ge 0$. Suppose that $0 \le x \le x' \le \bar{X}$. Let $\varepsilon > 0$ be arbitrary and

assume that $X \in \mathcal{A}_t(x)$ is an ε-optimal strategy that satisfies

$$v_i(t, x, y) + \varepsilon \geq \mathbb{E}[C_i(t, x, y, X)].$$

Let τ be an $\mathcal{F}$-stopping time, such that

$$\tau := \inf \left\{ u \geq t : X_u - x \geq \overline{X} - x' \right\}.$$

We define the strategy $X' \in \mathcal{A}_t(x')$ as

$$X'_u = \left\{ \begin{array}{ll} X_u + x' - x, & \text{if } u < \tau, \\ \overline{X}, & \text{else.} \end{array} \right.$$

Using the comparison property established in Lemma 1, we have that

$$C_i(t, x', y, X') \leq C_i(t, x, y, X).$$

Taking expectations on both sides, we deduce that

$$v_i(t, x', y) \leq \mathbb{E}[C_i(t, x', y, X')] \leq \mathbb{E}[C_i(t, x, y, X)] \leq v_i(t, x, y) + \varepsilon.$$

Since ε is arbitrary, the conclusion follows.

3. This result follows directly from the comparison property established for the process Y and the fact that Φ_i and ψ_i are non-decreasing on $\mathbb{R}_+$ for all $i \in \mathbb{I}_m$. $\qquad \square$

To prove the continuity of the value function, we need the following lemmata. The first result concerns the continuity of the flow of Y. The second is related to a continuity modulus.

Lemma 2. *For any non-decreasing, right-continuous, adapted process X and any initial time $t \in [0, T]$, the mapping $y \mapsto Y^{t,y,X}_\cdot$, representing the flow of the solution to SDE (4), is continuous on $\mathbb{R}_+$.*

Proof. See Theorems V.37 and V.38 in Protter [38]. $\qquad \square$

Lemma 3. *Let $0 \leq t \leq s \leq T$, $x \in [0, \overline{X}]$, $y \geq 0$ and $X \in \mathcal{A}_t(x)$. For any nonnegative random variable ξ with finite moments, it holds that*

$$\mathbb{E}\left[\Phi_{I_s}\left(Y^{t,y,X}_s + \xi \right) - \Phi_{I_s}\left(Y^{t,y,X}_s \right) \right] \leq \rho_y(\xi),$$

where the mapping $(y, \zeta) \mapsto \rho_y(\zeta)$ is continuous in y and non-decreasing in ζ on $\mathbb{R}_+$, satisfying $\lim_{\zeta \to 0} \rho_y(\zeta) = 0$. Additionally, $\rho_y(\xi)$ converges in probability to 0 as ξ tends to 0 in probability, if its moments are uniformly bounded.

Proof. Refer to Appendix A. $\qquad \square$

We conclude this section by presenting a key result on the continuity of the value function.

Theorem 3.1. *The value function v defined in (6) is a continuous function on $\overline{\mathcal{S}}$.*

Proof. We will study separately the continuity of the value function in t, x and y, for each $i \in \mathbb{I}_m$.

1. We first prove the continuity of v_i in x, uniformly with respect to t. We fix $t \in [0, T]$, $y \geq 0$, $0 \leq x' < x \leq \overline{X}$ and $\varepsilon > 0$. There exists $X \in \mathcal{A}_t(x)$ which satisfies

$$v_i(t, x, y) + \varepsilon \geq \mathbb{E}\left[C_i(t, x, y, X) \right].$$

By adding a jump of size $x - x'$ at time T to X we obtain $\widehat{X} \in \mathcal{A}(t, x')$. We have that

$$C_i(t, x', y, \widehat{X}) = C_i(t, x, y, X) + \Phi_{I_T}(Y^{t,y,X}_T + x - x') - \Phi_{I_T}(Y^{t,y,X}_T).$$

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Taking expectations on both sides of this equation we deduce that

$$\mathbb{E}\big[C_i(t, x', y, \widehat{X})\big] \leq v_i(t, x, y) + \varepsilon + \mathbb{E}\Big[\Phi_{I_T}(Y_T^{t,y,X} + x - x') - \Phi_{I_T}(Y_T^{t,y,X})\Big].$$

On the other hand since, by Proposition 3.2, the value function is decreasing in x, we conclude that

$$v_i(t, x, y) \leq v_i(t, x', y) \leq v_i(t, x, y) + \varepsilon + \mathbb{E}\Big[\Phi_{I_T}(Y_T^{t,y,X} + x - x') - \Phi_{I_T}(Y_T^{t,y,X})\Big].$$

We deduce from Lemma 3 that

$$0 \leq v_i(t, x', y) - v_i(t, x, y) \leq \varepsilon + \rho_y(x - x').$$

Therefore, we have obtained the continuity of the value function v in x uniformly in t.

2. Next, we prove the continuity of v_i in y, uniformly with respect to t and x. Fix $y \geq 0$, $y' \geq 0$, $\varepsilon > 0$, $t \in [0, T]$ and $x \in [0, \overline{X}]$. There exists $X \in \mathcal{A}_t(x)$ which satisfies

$$v_i(t, x, y) + \varepsilon \geq \mathbb{E}\big[C_i(t, x, y, X)\big].$$

Our goal here will be to construct an admissible strategy $\widetilde{X}$ that bounds

$$v_i(t, x, y') - v_i(t, x, y) \leq \varepsilon + \mathbb{E}\big[C_i(t, x, y', \widetilde{X}) - C_i(t, x, y, X)\big].$$

To do so, we compare the sample paths of $Y^{t,y,X}$ and $Y^{t,y',\widetilde{X}}$, and aim to bound their difference using terms that depend only on the initial parameters y, y' and their difference $|y' - y|$, as illustrated in Figure 2. This, combined with the continuity modulus presented in Lemma 3, should allow us to achieve the uniform continuity in y. Consider $\widehat{X} \in \mathcal{A}_t(x)$ such that

$$d\widehat{X}_u^c = 0, \quad \text{and} \quad \Delta\widehat{X}_u = \Delta X_u, \quad \forall u \in [0, T[.$$

Let τ be the following $\mathcal{F}$-stopping time,

$$\tau := \inf\big\{u \geq t : \ Y_u^{t,y',\widehat{X}} \leq Y_u^{t,y,X}\big\} \wedge T,$$

ξ define the random variable

$$\xi := \int_t^\tau dX_u^c - \big(Y_\tau^{t,y,X} - Y_\tau^{t,y',\widehat{X}}\big),$$

and θ be an $\mathcal{F}$-stopping time, such that

$$\theta := \inf\big\{u \geq \tau : \ \widehat{X}_\tau + X_u - X_\tau \geq \overline{X}\big\}.$$

Observe that when $y' \leq y$, $\tau = t$ and $\xi = y' - y \leq 0$. We construct a strategy $\widetilde{X}$ belonging to $\mathcal{A}_t(x)$ satisfying

$$d\widetilde{X}_u := \begin{cases} d\widehat{X}_u, & \text{if } t \leq u < \tau, \\ \big(\Delta X_u + Y_u^{t,y,X} - Y_u^{t,y',\widehat{X}}\big) \wedge \big(\overline{X} - \widehat{X}_{\tau-}\big), & \text{if } u = \tau, \\ dZ_u, & \text{if } \tau < u \leq T. \end{cases}$$

where Z is a non-decreasing $\mathcal{F}$-adapted process with $Z_\tau = 0$. If $\xi \geq 0$,

$$dZ_u := \begin{cases} dX_u, & \text{if } \tau < u < T, \\ \Delta X_u + \xi, & \text{if } u = T. \end{cases}$$

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If $\xi < 0$,

$$\mathrm{d}Z_u = \begin{cases} \mathrm{d}X_u, & \text{if } \tau < u < \theta, \\ \overline{X} - (\widehat{X}_\tau + X_{\theta^-} - X_\tau), & \text{if } u = \theta \text{ and } \tau < \theta \leq T, \\ 0, & \text{else }. \end{cases}$$

The goal of this construction is to align the paths of $u \mapsto Y_u^{t,y,X}$ and $u \mapsto Y_u^{t,y',\widehat{X}}$ at a chosen time τ, allowing for their comparison and the derivation of bounds for inequality (10).
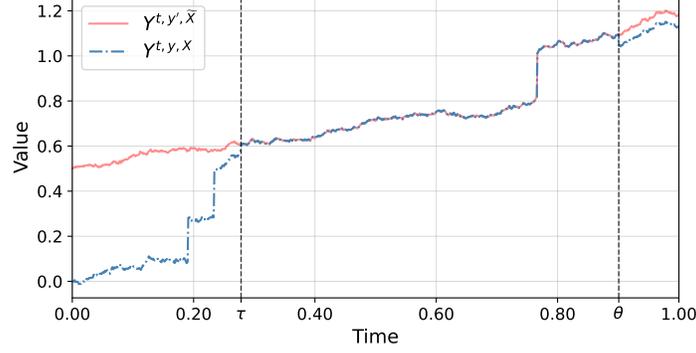

Figure 2: Illustration of $Y^{t,y,X}$ and $Y^{t,y',\widetilde{X}}$ sample paths in the scenario where $y < y'$, $\tau < \theta < T$ and $\Delta X_u + Y_u^{t,y,X} - Y_u^{t,y',\widehat{X}} < \overline{X} - \widehat{X}_{\tau^-}$.

Therefore, we obtain the following inequality

$$v_i(t,x,y') - v_i(t,x,y) \leq \varepsilon + R_1 + R_2 + R_3 + R_4 + R_5 + R_6, \tag{10}$$

where

$$R_1 = \mathbb{E}\Big[\int_t^T \psi_{I_{u^-}}(\check{Y}_{u^-}^{t,y',\widetilde{X}})\mathrm{d}\widetilde{X}_u^c - \int_t^T \psi_{I_{u^-}}(\check{Y}_{u^-}^{t,y,X})\mathrm{d}X_u^c\Big] \leq 0,$$

$$R_2 = \mathbb{E}\Big[\sum_{t \leq u < \tau; \Delta X_u > 0} \Phi_{I_u}(Y_u^{t,y',\widetilde{X}}) - \Phi_{I_{u^-}}(\check{Y}_{u^-}^{t,y',\widetilde{X}})\Big] - \mathbb{E}\Big[\sum_{t \leq u < \tau; \Delta X_u > 0} \Phi_{I_u}(Y_u^{t,y,X}) - \Phi_{I_{u^-}}(\check{Y}_{u^-}^{t,y,X})\Big],$$

$$R_3 = \mathbb{E}\Big[\Phi_{I_\tau}(Y_\tau^{t,y',\widetilde{X}}) - \Phi_{I_{\tau^-}}(\check{Y}_{\tau^-}^{t,y',\widetilde{X}})\Big] - \mathbb{E}\Big[\Phi_{I_\tau}(Y_\tau^{t,y,X}) - \Phi_{I_{\tau^-}}(\check{Y}_{\tau^-}^{t,y,X})\Big],$$

$$R_4 = \mathbb{E}\Big[\mathbb{1}_{\{\xi \geq 0\}}(\Phi_{I_T}(Y_{T^-}^{t,y',\widetilde{X}} + \xi) - \Phi_{I_T}(Y_{T^-}^{t,y,X}))\Big],$$

$$R_5 \leq \mathbb{E}\Big[\mathbb{1}_{\{\xi < 0\}}(\Phi_{I_\theta}(\check{Y}_{\theta^-}^{t,y',\widetilde{X}} + \Delta\widetilde{X}_\theta) - \Phi_{I_\theta}(\check{Y}_{\theta^-}^{t,y,X} + \Delta X_\theta))\Big] \leq 0.$$

We begin with finding an upper bound for R_2. Denote by $u_k \in [t,\tau[$ the jump times of X. Note that $y \mapsto \Phi_{I_{u^-}}(y)$ is non-decreasing on $\mathbb{R}$. An application of the comparison property of Y (see Lemma 1) yields $\check{Y}_{u^-}^{t,y,X} \leq \check{Y}_{u^-}^{t,y',\widetilde{X}} \leq \check{Y}_{u^-}^{t,y',X}$, for all $u < \tau$. Therefore,

$$\begin{aligned} R_2 &\leq \mathbb{E}\Big[\sum_{t \leq u_k < \tau} \Phi_{I_{u_k}}(Y_{u_k}^{t,y',X}) - \Phi_{I_{u_k}}(Y_{u_k}^{t,y,X})\Big] \\ &\leq \mathbb{E}\Big[\sum_{t \leq u_k < \tau} \Phi_{I_{u_k}}(Y_{u_k}^{t,y,X} + \gamma_{u_k}) - \Phi_{I_{u_k}}(Y_{u_k}^{t,y,X})\Big], \end{aligned} \tag{11}$$

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with $\gamma_u := \check{Y}_u^{t,y',X} - \check{Y}_u^{t,y,X} \geq 0$, for all $u \in [t,\tau[$. It follows from the dynamics of Y that

$$
\begin{aligned}
\mathrm{d}(\check{Y}_u^{t,y',X} - &\check{Y}_u^{t,y,X})^c \\
&= c(\check{Y}_{u^-}^{t,y',X} - \check{Y}_{u^-}^{t,y,X})^{c-1}\left[-\left(h(\check{Y}_{u^-}^{t,y',X}) - h(\check{Y}_{u^-}^{t,y,X})\right)\mathrm{d}u + \left(\sigma(\check{Y}_{u^-}^{t,y',X}) - \sigma(\check{Y}_{u^-}^{t,y,X})\right)\mathrm{d}W_u\right] \\
&\quad + \frac{c(c-1)}{2}(\check{Y}_{u^-}^{t,y',X} - \check{Y}_{u^-}^{t,y,X})^{c-2}\left|\sigma(\check{Y}_{u^-}^{t,y',X}) - \sigma(\check{Y}_{u^-}^{t,y,X})\right|^2 \mathrm{d}u \\
&\quad + \int_{\mathbb{R}}\left[\left(\check{Y}_{u^-}^{t,y',X} - \check{Y}_{u^-}^{t,y,X} + q(\check{Y}_{u^-}^{t,y',X}, z) - q(\check{Y}_{u^-}^{t,y,X}, z)\right)^c - \left(\check{Y}_{u^-}^{t,y',X} - \check{Y}_{u^-}^{t,y,X}\right)^c\right]M(\mathrm{d}u, \mathrm{d}z) \\
&\quad + \int_{\mathbb{R}} c(\check{Y}_{u^-}^{t,y',X} - \check{Y}_{u^-}^{t,y,X})^{c-1}\left(q(\check{Y}_{u^-}^{t,y',X}, z) - q(\check{Y}_{u^-}^{t,y,X}, z)\right)\lambda_u\nu(\mathrm{d}z)\,\mathrm{d}u,
\end{aligned}
$$

for all $u \in [t,T]$ and $c > 0$. Using Assumption (A4) that for all jump times $u_k \in [t,\tau[$,

$$
\mathbb{E}[|\gamma_{u_k}|^c] \leq |y' - y| + L'\mathbb{E}\left[\int_0^{u_k}|Y_{u^-}^{t,y,X} - Y_{u^-}^{t,y',X}|^c\,\mathrm{d}u\right].
$$

with $L' > 0$. Thanks to the Grönwall inequality, we can then assert that

$$
\mathbb{E}[|\gamma_{u_k}|^c] \leq |y' - y|e^{LT}.
$$

Applying the inequality $(1+x)^\gamma - 1 \leq \gamma x + C_\gamma x^\gamma$ with $\gamma = \frac{\beta+1}{\beta} > 1$ and $C_\gamma := \sup_{x>0}\frac{(1+x)^\gamma - 1 - \gamma x}{x^\gamma} \geq 0$ for $x > -1$ (see Hardy, Littlewood, and Pólya [28, Chapter 3]), we obtain

$$
\rho_y(\xi) = \sum_{i=1}^m C_i^y \mathbb{E}\left[\left(\left(1 + \frac{\xi}{F_i(a)}\right)^\gamma - 1\right)^2\right]^{\frac{1}{2}} \leq \sum_{i=1}^m C_i^y\left(\frac{2\gamma^2\,\mathbb{E}[\xi^2]}{F_i(a)^2} + \frac{2C_\gamma^2\,\mathbb{E}[|\xi|^{2\gamma}]}{F_i(a)^{2\gamma}}\right)^{\frac{1}{2}}. \tag{12}
$$

Lemma 3 combined with (11) and iterated expectations yields

$$
R_2 \leq \mathbb{E}\left[\sum_{t \leq u_k < \tau}\rho_y(\gamma_{u_k})\right] \leq T\rho_y'(|y' - y|),
$$

where $\rho_y' : \xi \mapsto \sum_{i=1}^m C_i^y\left(\frac{2\gamma^2\,\mathbb{E}[\xi^2]}{F_i(a)^2} + \frac{2C_\gamma^2\,\mathbb{E}[|\xi|^{2\gamma}]}{F_i(a)^{2\gamma}}\right)^{\frac{1}{2}}$. Similarly, we have that

$$
R_3 \leq \rho_y'(|y' - y|)
$$

since $y \mapsto \Phi_{I_{u^-}}(y)$ is non-decreasing on $\mathbb{R}$, and $\check{Y}_{\tau^-}^{t,y,X} \leq \check{Y}_{\tau^-}^{t,y',\widetilde{X}} \leq \check{Y}_{\tau^-}^{t,y',X}$, for $u < \tau$. The second inequality follows directly from Lemma 1. We turn to finding an upper bound for R_4. Notice that we have $R_5 = 0$ when $y' \leq y$. If $y' > y$, then $\mathbb{E}[\xi] \leq y' - y$. In this case,

$$
R_4 \leq \mathbb{E}\left[\Phi_{I_T}(Y_T^{t,y,X} + \gamma_T + \xi) - \Phi_{I_T}(Y_T^{t,y,X})\right].
$$

Observe that ξ satisfies the conditions of Lemma 3 due to the strong solution result in Proposition 2.1, and that the admissible controls have bounded variations. Therefore,

$$
R_4 \leq \mathbb{E}\left[\rho_y(\gamma_\tau + \gamma_T)\right] \leq \rho_y'(2|y' - y|).
$$

We conclude this proof by recalling that $\zeta \to \rho_y(\zeta)$ is uniformly continuous on $\mathbb{R}_+$, for all $y \geq 0$.

3. Finally, we demonstrate the continuity of v in t. Let $x \in [0, \overline{X}]$, $y \geq 0$ and $0 \leq u < t$. Now, consider a strategy $X \in \mathcal{A}_u(x)$ such that

$$
v_i(u, x, y) + \varepsilon \geq C_i(u, x, y, X).
$$

In the following, we will construct a strategy $\widehat{X} \in \mathcal{A}_s(x)$ that consists of not doing anything on $[s, t[$ and then following the strategy X similarly to the previous proof. We introduce the $\mathcal{F}$-stopping time τ, defined by

$$\tau := \inf\left\{u \geq t : \ \check{Y}_{u^-}^{u,y,\widehat{X}} \leq y\right\} \wedge T.$$

Let $\widehat{X}$ define an admissible strategy in $\mathcal{A}_t(x)$, such that

$$\mathrm{d}\widehat{X}_u = \begin{cases} \left(\Delta X_\tau + y - \check{Y}_{\tau^-}^{u,y,\widehat{X}}\right) \wedge (\overline{X} - x), & \text{if } u = \tau, \\ \mathrm{d}X_u, & \text{if } \tau < u < \theta, \\ \overline{X} - \widehat{X}_{u^-}, & \text{if } u = \theta, \\ 0, & \text{if } \theta < u \leq T, \end{cases}$$

where

$$\theta := \inf\left\{u \geq \tau : \ \widehat{X}_{u^-} + \Delta X_u \geq \overline{X}\right\} \wedge T.$$

Considering an ε-optimal strategy $X \in \mathcal{A}_t(x)$, we can show that

$$\begin{aligned} v_i(u, x, y) - v_i(t, x, y) &\leq C_i(u, x, y, \widehat{X}) - C_i(t, x, y, X) + \varepsilon \\ &\leq \mathbb{E}\left[C_i(t, x, Y_{t^-}^{u,y,\widehat{X}}, \widehat{X}) - C_i(t, x, y, X)\right] + \varepsilon \\ &\leq f(\varepsilon), \end{aligned}$$

with $\lim_{\varepsilon \to 0} f(\varepsilon) = 0$.

We can combine the uniform continuity of v_i in x with respect to t, in y uniformly with respect to t and x, and in t to get joint continuity on $\overline{\mathcal{S}}$. $\qquad\square$

Proposition 3.3. *The value function v defined in* (6) *is differentiable almost everywhere on $\overline{\mathcal{S}}$ and has finite one-derivatives everywhere.*

Proof. The value function v_i is continuous and monotonic on $\overline{\mathcal{S}}$, and thus, by Lebesgue's Theorem, it is differentiable almost everywhere for all $i \in \mathbb{I}_m$. Therefore, there exists a set of Lebesgue measure zero where v_i may not be differentiable. We denote the right and left partial derivatives of v_i with respect to x and y as $\partial_x^\pm v_i$ and $\partial_y^\pm v_i$. Based on the proof of Theorem 3.1, we have that

$$0 \leq \frac{v_i(t, x+h, y) - v_i(t, x, y)}{h} \leq \frac{\rho_y'(h)}{h}, \quad \forall(t, x, y, h) \in \overline{\mathcal{S}} \times \mathbb{R}_+.$$

Moreover, it follows that

$$\lim_{h \to 0^+} \frac{\rho_y'(h)}{h} = \sum_{i=1}^m C_i^y \frac{\sqrt{2}\gamma}{F_i(a)} < +\infty,$$

which implies that $\partial_x^+ v_i$ is finite on $\overline{\mathcal{S}}$. Similarly, $\partial_x^- v_i$ is finite. Hence, v_i is differentiable in x on $\overline{\mathcal{S}}$. The same argument applies to t and y, establishing differentiability in all variables on $\overline{\mathcal{S}}$. $\qquad\square$

4 Viscosity Characterization of the Value Function

This section aims to provide a PDE characterization of the value functions v.

Lemma 4 (Dynamic programming principle). *For any stopping time τ in $[t, T]$, we have*

$$v_i(t, x, y) =$$
$$\inf_{X \in \mathcal{A}_t(x)} \mathbb{E}\left[\int_t^\tau \psi_{I_{u^-}}(\check{Y}_{u^-}^{t,y,X})\mathrm{d}X_u^c + \sum_{t \leq u \leq \tau}\left(\Phi_{I_u}(Y_u^{t,y,X}) - \Phi_{I_{u^-}}(\check{Y}_{u^-}^{t,y,X})\right) + v(\tau, I_\tau, X_\tau, Y_\tau^{t,y,X})\right],$$

where $(t, x, y) \in \mathcal{S}$ and $i \in \mathbb{I}_m$.

Proof. This is a straightforward adaptation of the results from Bouchard and Touzi [10], leveraging the continuity of the value functions v_i established in Theorem 3.1. $\qquad\square$

Building on the problem formulation 2.2, we consider the following Hamilton-Jacobi-Bellman Quasi-Variational Inequalities for v, for all $i \in \mathbb{I}_m$,

$$\max\big(-\frac{\partial v_i}{\partial t} - \mathcal{L}v_i - \sum_{j\neq i}(v_j - v_i)Q_{ij}, -\frac{\partial v_i}{\partial x} - \frac{\partial v_i}{\partial y} - \psi_i\big) = 0, \text{ on } \mathcal{S}. \tag{13}$$

The partial integro-differential operator $\mathcal{L}$ is given by

$$\mathcal{L}\varphi := \frac{1}{2}\sigma^2(y)\frac{\partial^2\varphi}{\partial y^2} - h(y)\frac{\partial\varphi}{\partial y} + \lambda_t \int_{\mathbb{R}} (\varphi(t, x, y + q(y, z)) - \varphi)\,\nu(\mathrm{d}z).$$

The boundary/terminal conditions here are specified in (7).

Remark 4.1. *Vectors are assumed to be m-dimensional unless stated otherwise, and all operations are performed component-wise. Deviations from these notations must be explicitly specified.*

4.1 Viscosity Solution Property

As the value function v may lack smoothness, it is formulated as a discontinuous viscosity solution of a quasi-variational inequalities (13) with appropriate boundary conditions, following Crandall, Ishii, and Lions [21].

Definition 4.1 (Viscosity solution)**.** *We define a viscosity solution of (13) as follows :*

1. *v is a continuous viscosity supersolution (resp. subsolution) of (13) on $\mathbb{I}_m \times \mathcal{S}$ if it satisfies the growth conditions (8), and if*

$$\max\Big(-\big(\frac{\partial\varphi}{\partial t} + \mathcal{L}\varphi + \sum_{j\neq i}(v_j - v_i)Q_{ij}\big)(t, x, y), -\big(\frac{\partial\varphi}{\partial x} + \frac{\partial\varphi}{\partial y} + \psi_i\big)(t, x, y)\Big) \geq 0\,(\text{resp. } \leq),\ (14)$$

   for any $(i, t, x, y) \in \mathbb{I}_m \times \mathcal{S}$ and any smooth test function $\varphi \in C^{1,2}(\mathcal{S})$ such that $(v_i - \varphi)$ attains a local minimum (resp. maximum) at (t, x, y) over the set $[t, t + \delta[\times[x, x + \delta[\times B_\delta(y) \subset \mathcal{S}$ for some $\delta > 0$, with $(v_i - \varphi)(t, x, y) = 0$.

2. *v is a continuous viscosity solution on $\mathbb{I}_m \times \mathcal{S}$ if it is both a viscosity supersolution and subsolution of (13).*

Next, we prove that the value function is a viscosity solution as defined in Definition 4.1.

Theorem 4.1. *The value function v defined in (6) is a viscosity subsolution of (13).*

Proof. Refer to Appendix B.1. $\qquad\square$

Theorem 4.2. *The value function v defined in (6) is a viscosity supersolution of (13).*

Proof. Refer to Appendix B.2. $\qquad\square$

4.2 Uniqueness Result

The continuity of v established in Theorem 3.1, together with the results of Theorems 4.1 and 4.2, and the at most polynomial growth of the power function, show that v is a viscosity solution of (13). Although the techniques used to prove the comparison theorem are standard, the regime switching and jump terms render it nontrivial, motivating us to present the full argument here.

Theorem 4.3 (Strong comparison principle). *If v_i is a continuous viscosity subsolution of* (14) *and w_i is a continuous viscosity supersolution of* (14)*, such that*

$$v_i(t, \overline{X}, y) \leq w_i(t, \overline{X}, y), \quad and \quad v_i(T, x, y) \leq w_i(T, x, y),$$

for all $(i, t, x, y) \in \mathbb{I}_m \times \overline{\mathcal{S}}$, then $v_i \leq w_i$ on $\mathcal{S}$.

Proof. Let $\beta > 0$ such that

$$\lim_{y \to +\infty} \max_{i \in I_m} \frac{\Phi_i(y + \overline{X} - x) - \Phi_i(y)}{y^\beta} = 0, \quad \forall (x, y) \in [0, \overline{X}] \times \mathbb{R}_+. \tag{15}$$

The existence of such a constant β follows from Assumption (A1). Let $i \in \mathbb{I}_m$, and define $\varphi_i : \overline{\mathcal{S}} \to \mathbb{R}$ such that

$$\varphi_i(t, x, y) := -e^{-ct}\big((-a_1 x + a_2) y^\beta - b_1 x + b_2\big), \quad \forall (t, x, y) \in \overline{\mathcal{S}},$$

where a_1, a_2, b_1, b_2 and c are positive constants satisfying $a_2 > a_1 \overline{X}$ and $b_2 > b_1 \overline{X}$. Additionally, we set $a(x) := -a_1 x + a_2 > 0$, for all $x \in [0, \overline{X}]$. Fix an arbitrary point $z := (t, x, y) \in \overline{\mathcal{S}}$. Hence, we obtain the following inequality

$$\big(\frac{\partial \varphi_i}{\partial t} + \mathcal{L}\varphi_i\big)(z) = e^{-ct}\big(c(a(x)y^\beta - b_1 x + b_2) - \frac{\beta(\beta - 1)}{2} a(x)\sigma(y)^2 y^{\beta - 2}\big)$$

$$+ e^{-ct}\big(-\beta h(y)a(x)y^{\beta - 1} + \lambda_t a(x) \int_{\mathbb{R}_+} ((y + q(y, z))^\beta - y^\beta)\,\nu(\mathrm{d}z)\big)$$

$$\geq a(x)e^{-ct}y^\beta\big(c - \beta \frac{h(y)}{y} - \frac{\beta(\beta - 1)}{2}\frac{\sigma(y)^2}{y^2} - \bar{\lambda}\int_{\mathbb{R}_+}\nu(\mathrm{d}z)\big) + ce^{-cT}\big(-b_1\overline{X} + b_2\big)$$

$$\geq ce^{-cT}\big(-b_1\overline{X} + b_2\big).$$

The final passage follows from the linear growth assumption (A3) on h and σ and is true for c big enough. Moreover, knowing that $\psi_i \geq 0$, we have

$$\big(\frac{\partial \varphi_i}{\partial x} + \frac{\partial \varphi_i}{\partial y} + \psi_i\big)(z) = e^{-ct}\big(a_1 y^\beta + b_1 - \beta a(x)y^{\beta - 1}\big) + \psi_i(y) \geq \varepsilon,$$

for $\varepsilon > 0$ and b_1 big enough compared to a_1 and a_2. Define $v_{i,m} : \overline{\mathcal{S}} \to \mathbb{R}$ such that

$$v_{i,m} := v_i + \frac{1}{m}\varphi_i,$$

with $m \in \mathbb{N}^*$. Combining (15) with the continuity of φ_i on $\overline{\mathcal{S}}$, we get that

$$\lim_{y \to +\infty} \frac{1}{m}\varphi_i(t, x, y) + \Phi_i(y + \overline{X} - x) - \Phi_i(y) = -\infty, \quad \forall (t, x) \in [0, T] \times [0, \overline{X}].$$

Note that $\sum_{j \neq i}(\varphi_j - \varphi_i)Q_{ij} = 0$. Therefore, $v_{i,m}$ is a strict subsolution of equation (14) in the sense that it satisfies

$$\max\big(-\frac{\partial v_{i,m}}{\partial t} - \mathcal{L}v_{i,m} - \sum_{j \neq i}(\varphi_j - \varphi_i)Q_{ij},\ -\frac{\partial v_{i,m}}{\partial x} - \frac{\partial v_{i,m}}{\partial y} - \psi_i\big) \leq -\frac{\varepsilon}{m} < 0, \quad \text{on } \mathcal{S}. \tag{16}$$

Toward Ishii's lemma. Our goal is to show that $\varrho := \sup_{z \in S} v_{i,m}(z) - w_i(z) \leq 0$. Suppose on the contrary that $\varrho > 0$. Based on the growth condition (8), we know that

$$v_i(z) \leq \Phi_i(y + \overline{X} - x) - \Phi_i(y), \quad \forall z \in \mathcal{S}.$$

Therefore,
$$\lim_{x \to \overline{X}} v_{i,m} - w_i \leq 0.$$

Additionally, using the previous results, we get,
$$\lim_{y \to +\infty} v_{i,m} - w_i = -\infty, \text{ and } \lim_{t \to T} v_{i,m} - w_i \leq 0.$$

This means that the supremum is attained an interior point $z_0 := (t_0, x_0, y_0) \in \overline{\mathcal{O}} \subset \mathcal{S}$, with $\mathcal{O} :=]0, T[\times]0, \overline{X}[\times]0, \overline{Y}[$. In other words, we have $\varrho = v_{i,m}(z_0) - w_i(z_0)$. Let $k \geq 1$ and define, for all $(t, x, y, y') \in \mathcal{S} \times \mathbb{R}_+$,
$$H_k(t, x, y, y') := v_{i,m}(t, x, y) - w_i(t, x, y') - d_k(y, y'),$$

where $d_k(y, y') := \frac{k}{2}|y - y'|^2$. Define $\varrho_k := \sup_{\overline{\mathcal{O}} \times [0, \overline{Y}]} H_k(t, x, y, y')$. Since H_k is continuous and coercive, its supremum is attained at some point $(\hat{t}_k, \hat{x}_k, \hat{y}_k, \hat{y}'_k) \in \overline{\mathcal{O}} \times [0, \overline{Y}]$. By means of the Bolzano–Weierstrass theorem, there exists a subsequence $(\hat{t}_{n_k}, \hat{x}_{n_k}, \hat{y}_{n_k}, \hat{y}'_{n_k})$ that converges to a point $(\hat{t}_0, \hat{x}_0, \hat{y}_0, \hat{y}'_0) \in \overline{\mathcal{O}} \times [0, \overline{Y}]$ as $k \to +\infty$. In the following, we will continue using k as an index instead of n_k to avoid the proliferation of indices. For k large enough, we can then assume that $\hat{t}_k < T$, and $\hat{x}_k > 0$. Consider the following inequality
$$H_k(\hat{t}_0, \hat{x}_0, \hat{y}_0, \hat{y}'_0) \leq H_k(\hat{t}_k, \hat{x}_k, \hat{y}_k, \hat{y}'_k).$$

In particular, we have
$$\frac{k}{2}|\hat{y}_k - \hat{y}'_k|^2 \leq -v_{i,m}(\hat{t}_0, \hat{x}_0, \hat{y}_0) + w_i(\hat{t}_0, \hat{x}_0, \hat{y}_0) + v_{i,m}(\hat{t}_k, \hat{x}_k, \hat{y}_k) - w_i(\hat{t}_k, \hat{x}_k, \hat{y}'_k).$$

As $v_{i,m}$ and w_i are continuous on the compact set $\overline{\mathcal{O}}$, there exists $C > 0$ such that
$$|\hat{y}_k - \hat{y}'_k|^2 \leq \frac{C}{k}. \tag{17}$$

Letting k go to $+\infty$, we find $\hat{y}_0 = \hat{y}'_0$. Finally, we show that ϱ_k tends to ϱ when k goes to $+\infty$. Note that
$$\varrho = v_{i,m}(z_0) - w_i(z_0) = H_k(t_0, x_0, y_0, y_0) \leq H_k(\hat{t}_k, \hat{x}_k, \hat{y}_k, \hat{y}'_k).$$

Therefore, $\varrho \leq \varrho_k$. Moreover, we have
$$\varrho_k = v_{i,m}(\hat{t}_k, \hat{x}_k, \hat{y}_k) - w_i(\hat{t}_k, \hat{x}_k, \hat{y}'_k) - \frac{k}{2}|\hat{y}_k - \hat{y}'_k|^2 \leq v_{i,m}(\hat{t}_k, \hat{x}_k, \hat{y}_k) - w_i(\hat{t}_k, \hat{x}_k, \hat{y}'_k).$$

Since $v_{i,m}$ and w_i are continuous on $\mathcal{S}$, we get that
$$\lim_{k \to +\infty} v_{i,m}(\hat{t}_k, \hat{x}_k, \hat{y}_k) - w_i(\hat{t}_k, \hat{x}_k, \hat{y}'_k) = v_{i,m}(\hat{t}_0, \hat{x}_0, \hat{y}_0) - w_i(\hat{t}_0, \hat{x}_0, \hat{y}_0) \leq \varrho.$$

We conclude that $\lim_{k \to +\infty} \varrho_k = \varrho$ and $\lim_{k \to +\infty} |\hat{y}_k - \hat{y}'_k|^2 = 0$. Moreover, we have
$$v_{i,m}(\hat{t}_0, \hat{x}_0, \hat{y}_0) - w_i(\hat{t}_0, \hat{x}_0, \hat{y}_0) = \varrho.$$

Ishii's lemma. We now apply Theorem 3.2 from Crandall, Ishii, and Lions [21] at the point $(\hat{t}_k, \hat{x}_k, \hat{y}_k, \hat{y}'_k)$, ensuring the existence of $M, M' \in \mathbb{R}$ such that
$$\begin{pmatrix} -k - \|A\| & 0 \\ 0 & -k - \|A\| \end{pmatrix} \leq \begin{pmatrix} M & 0 \\ 0 & M' \end{pmatrix} \leq A + \frac{1}{k}A^2,$$

with $A = D^2 d_k(\hat{y}_k, \hat{y}'_k) = \begin{pmatrix} k & -k \\ -k & k \end{pmatrix}$. Let $\mathcal{K}$ and $\mathcal{I}$ be the operators defined by

$$\mathcal{K}[p, M](y) := \frac{\sigma^2(y)}{2} M - h(y)p, \text{ and } \mathcal{I}[g](t, x, y) := \int_{\mathbb{R}} (g(t, x, y + q(y, \zeta)) - g(t, x, y))\nu(\mathrm{d}\zeta). \quad (18)$$

Using the relationship between superjets and Definition 4.1 of viscosity supersolutions, along with inequality (16), we deduce from Ishii's Lemma that

$$\varepsilon/m \leq \min\left(\mathcal{K}[\frac{\partial d_k}{\partial y}(\hat{y}_k, \hat{y}'_k), M](\hat{y}_k) + \lambda_{\hat{t}_k} \mathcal{I}[v_{i,m}](\hat{t}_k, \hat{x}_k, \hat{y}_k), \ \frac{\partial d_k}{\partial y}(\hat{y}_k, \hat{y}'_k) + \psi_i(\hat{y}_k) \right), \quad (19)$$

$$0 \geq \min\left(-\mathcal{K}[\frac{\partial d_k}{\partial y'}(\hat{y}_k, \hat{y}'_k), M'](\hat{y}'_k) + \lambda_{\hat{t}_k} \mathcal{I}[w_i](\hat{t}_k, \hat{x}_k, \hat{y}'_k), \ -\frac{\partial d_k}{\partial y'}(\hat{y}_k, \hat{y}'_k) + \psi_i(\hat{y}'_k) \right). \quad (20)$$

From the second inequality, we have two cases:

1. $-\frac{\partial d_k}{\partial y'}(\hat{y}_k, \hat{y}'_k) + \psi_i(\hat{y}'_k) \leq 0.$

2. $-\mathcal{K}[\frac{\partial d_k}{\partial y'}, M'](\hat{y}_k, \hat{y}'_k) + \lambda_{\hat{t}_k} \mathcal{I}[w_i](\hat{t}_k, \hat{x}_k, \hat{y}'_k) \leq 0.$

In the first case, we have $\psi_i(\hat{y}'_k) \leq \frac{\partial d_k}{\partial y'}(\hat{y}_k, \hat{y}'_k)$. Applying inequality (19), it follows that

$$\begin{aligned}
\frac{\varepsilon}{m} &\leq \frac{\partial d_k}{\partial y}(\hat{y}_k, \hat{y}'_k) + \psi_i(\hat{y}_k) - \psi_i(\hat{y}'_k) + \psi_i(\hat{y}'_k) \\
&\leq \frac{\partial d_k}{\partial y}(\hat{y}_k, \hat{y}'_k) + \frac{\partial d_k}{\partial y'}(\hat{y}_k, \hat{y}'_k) + \psi_i(\hat{y}_k) - \psi_i(\hat{y}'_k) \\
&\leq \psi_i(\hat{y}_k) - \psi_i(\hat{y}'_k).
\end{aligned}$$

Since ψ_i is continuous at $\hat{y}_0$, it follows that $\lim_{k \to +\infty} \psi_i(\hat{y}_k) - \psi_i(\hat{y}'_k) = 0$, which leads to a contradiction. In the second case, applying inequality (19) once more yields

$$\begin{aligned}
\frac{1}{2}(M\sigma(\hat{y}_k)^2 &+ M'\sigma(\hat{y}'_k)^2) - k(h(\hat{y}_k) - h(\hat{y}'_k))(\hat{y}_k - \hat{y}'_k) + \lambda_{\hat{t}_k}\left(\mathcal{I}[v_{i,m}](\hat{t}_k, \hat{x}_k, \hat{y}_k) - \mathcal{I}[w_i](\hat{t}_k, \hat{x}_k, \hat{y}'_k) \right) \\
&= \mathcal{K}[\frac{\partial d_k}{\partial y}(\hat{y}_k, \hat{y}'_k), M](\hat{y}_k) + \lambda_{\hat{t}_k} \mathcal{I}[v_{i,m}](\hat{t}_k, \hat{x}_k, \hat{y}_k) + \mathcal{K}[\frac{\partial d_k}{\partial y'}(\hat{y}_k, \hat{y}'_k), M'](\hat{y}'_k) - \lambda_{\hat{t}_k} \mathcal{I}[w_i](\hat{t}_k, \hat{x}_k, \hat{y}'_k) \\
&\geq \frac{\varepsilon}{m}.
\end{aligned}$$

Using the continuity of h, λ, $v_{i,m}$ and w_i, we deduce that

$$\begin{aligned}
\frac{\varepsilon}{m} &\leq \lambda_{\hat{t}_0}\left(\mathcal{I}[v_{i,m}] - \mathcal{I}[w_i] \right)(\hat{t}_0, \hat{x}_0, \hat{y}'_0) + \lim_{k \to +\infty} \frac{1}{2}\left\langle \begin{pmatrix} M & 0 \\ 0 & M' \end{pmatrix} (\sigma(\hat{y}_k), \sigma(\hat{y}'_k))^T, (\sigma(\hat{y}_k), \sigma(\hat{y}'_k)) \right\rangle \\
&\leq \lambda_{\hat{t}_0} \mathcal{I}[H_0](\hat{t}_0, \hat{x}_0, \hat{y}_0) + \lim_{k \to +\infty} \frac{1}{2}\left\langle (A + \frac{1}{k}A^2)(\sigma(\hat{y}_k), \sigma(\hat{y}'_k))^T, (\sigma(\hat{y}_k), \sigma(\hat{y}'_k)) \right\rangle,
\end{aligned}$$

with $H_0(t, x, y) := H_0(t, x, y, y)$, for all $(t, x, y) \in \mathcal{S}$. By definition of $\rho = H_0(\hat{t}_0, \hat{x}_0, \hat{y}_0)$, we get

$$H_0(\hat{t}_0, \hat{x}_0, \hat{y}_0 + q(y_0, \zeta)) - H_0(\hat{t}_0, \hat{x}_0, \hat{y}_0) \leq 0, \quad \forall \zeta \in \mathbb{R}.$$

Consequently, it follows that $\mathcal{I}[H_0](\hat{t}_0, \hat{x}_0, \hat{y}_0) \leq 0$ and

$$\frac{\varepsilon}{m} \leq \lim_{k \to +\infty} \frac{1}{2}\left\langle (A + \frac{1}{k}A^2)(\sigma(\hat{y}_k), \sigma(\hat{y}'_k))^T, (\sigma(\hat{y}_k), \sigma(\hat{y}'_k)) \right\rangle = \lim_{k \to +\infty} \frac{3}{2}k(\sigma(\hat{y}_k) - \sigma(\hat{y}'_k))^2 \leq \frac{3}{2}C.$$

The last equality follows from the Lipschitz continuity of σ and inequality (17). Since ε is arbitrary, this results in a contradiction, thereby concluding the proof. $\qquad\square$

4.3 The Free Boundary Problem

The value function v_i is differentiable almost everywhere by Proposition 3.3. Let $\mathcal{N}_i$ denote the set of Lebesgue measure zero where v_i may not be differentiable. We now define the associated free boundary.

Definition 4.2. *We define the exercise region $\mathcal{E}_i := \overline{int(\mathcal{E}_i^{diff})}$ as the closure of the interior of the set $\mathcal{E}_i^{diff}$, with*

$$\mathcal{E}_i^{diff} := \left\{ (t, x, y) \in \mathcal{S} \setminus \mathcal{N}_i : -\frac{\partial v_i}{\partial x} - \frac{\partial v_i}{\partial y} - \psi_i = 0 \right\},$$

and we define the continuation region $\mathcal{C}_i := \bar{\mathcal{S}} \setminus \mathcal{E}_i$ as its complement.

We now study the geometry of the free boundary $\partial \mathcal{E}_i := \bar{\mathcal{C}}_i \cap \mathcal{E}_i$ separating the continuation region $\mathcal{C}_i$ and the exercise region $\mathcal{E}_i$.

Lemma 5 (Partial smooth-fit principle). *For all $(t_0, x_0, y_0) \in \overline{\mathcal{S}} \setminus \mathcal{N}_i$, the mappings*

$$y \mapsto \partial_x v_i(t_0, x_0, y), \quad and \quad x \mapsto \partial_y v_i(t_0, x, 0).$$

are continuous almost everywhere on $\mathbb{R}_+$ and on the interval $[0, \overline{X}]$, respectively.

Proof. We know that v_i is jointly continuous on $\bar{\mathcal{S}}$ and uniformly continuous in the variable y. Additionally, the map $x \mapsto v_i(t, x, y)$ is differentiable for almost each $x \in [0, \bar{X}]$. Define, for $h \neq 0$,

$$D_h(t, x, y) := \frac{v_i(t, x + h, y) - v_i(t, x, y)}{h}, \quad \forall z \in \bar{\mathcal{S}} \setminus \mathcal{N}_i.$$

Based on the proof of Theorem 3.1, we have that

$$|D_h(t, x, y) - D_h(t, x, y')| \leq \frac{w(|y - y'|)}{|h|},$$

with w a continuous function on $\mathbb{R}_+$. Hence, the family $\{D_h\}_h$ is equicontinuous in y on compact subsets of $\bar{\mathcal{S}}$. Given the pointwise convergence $D_h \underset{h \to 0}{\to} \partial_x v_i$ almost everywhere, the equicontinuity and uniform boundedness in y, the Arzelà-Ascoli Theorem ensures uniform convergence in y over compact subsets of $\bar{\mathcal{S}}$, for almost every $x \in [0, \overline{X}]$. Hence, $\partial_x v_i$ is the uniform limit of continuous functions in y and $y \mapsto \partial_x v_i(t, x, y)$ is continuous almost everywhere on $\mathbb{R}_+$.

Similarly, the map $x \mapsto \partial_y v_i(t, x, 0)$ is continuous almost everywhere on $[0, \overline{X}]$. This follows from a combination of Rademacher's Theorem and the fact that the difference quotients for $(t, x) \mapsto \partial_y v_i(t, x, 0)$ converge uniformly in x on compact sets due to the joint continuity of v_i and uniform bound in x with respect to t. This concludes the proof. $\square$

Proposition 4.1 (Connectedness). *Assume that ψ_i is continuous, that the interiors of $\mathcal{C}_i$ and $\mathcal{E}_i$ are non-empty, and that $\sigma + \lambda > 0$. Then, for each $i \in \mathbb{I}_m$, the free boundary $\partial \mathcal{E}_i$ is non-empty and path-connected. Moreover, if $(t, x, y) \in \mathcal{E}_i$, then*

$$(t, x, y') \in \mathcal{E}_i \quad for\ all\ 0 \leq y' \leq y, \quad and \quad (t, x', 0) \in \mathcal{E}_i \quad for\ all\ 0 \leq x' \leq x.$$

Proof. As $\mathring{\mathcal{C}}_i$ and $\mathring{\mathcal{E}}_i$ are non-empty, we can consider $z_0 := (t_0, x_0, y_0) \in \mathring{\mathcal{E}}_i$ be an interior point of $\mathcal{E}_i$. By contradiction, we suppose that $\{0 < y < y_0 : (t_0, x_0, y) \in \mathcal{C}_i\}$ is non-empty. Being interior points, each admits a neighborhood fully contained in $\mathcal{C}_i$ or $\mathcal{E}_i$, respectively. Define the boundary point $z_1 := (t_0, x_0, y_1)$ with $y_1 := y_0 - \delta$. Let $\delta := \sup\{0 < y < y_0 : (t_0, x_0, y) \in \mathcal{C}_i\}$ and $\delta' > 0$ such that $\{(t_0, x_0, y) : y_0 - \delta - \delta' < y < y_0 - \delta\} \subset \mathcal{C}_i$. Define $\varepsilon := \min\{\delta, \delta'\}$. The following holds,

$$\mathcal{O}_c := \{(t_0, x_0, y) : y_0 - \delta - \varepsilon < y < y_0 - \delta\} \subset \mathcal{C}_i, \quad and \quad \mathcal{O}_e := \{(t_0, x_0, y) : y_0 - \delta < y < y_0 - \delta + \varepsilon\} \subset \mathcal{E}_i.$$

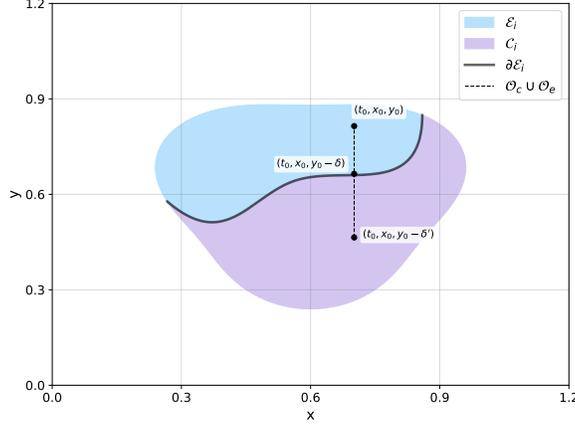Figure 3 provides a graphical representation of the previously defined sets.

Figure 3: Illustration of $\mathcal{O}_e$ and $\mathcal{O}_c$.

In this setting, define the function $f_0 : \mathcal{O}_c \cup \mathcal{O}_e \cup \{(t_0, x_0, y_1)\} \to \mathbb{R}$ satisfying

$$f_0(t_0, x_0, y) := \begin{cases} -\partial_x^+ v_i(t_0, x_0, y_1) - \partial_y^- v_i(t_0, x_0, y) - \psi_i(y), & \text{if } y < y_1. \\ -\partial_x^+ v_i(t_0, x_0, y_1) - \partial_y^+ v_i(t_0, x_0, y) - \psi_i(y), & \text{if } y \geq y_1. \end{cases} \qquad (21)$$

Given that $f_0(t_0, x_0, y_1) = \liminf_{y \to y_1^+} f_0(t_0, x_0, y) = 0$, two scenarios may arise:

$$\liminf_{y \to y_1^-} f_0(t_0, x_0, y) < \eta_0 < 0, \quad \text{or} \quad \liminf_{y \to y_1^-} f_0(t_0, x_0, y) = 0.$$

Case 1. $\liminf_{y \to y_1^-} f_0(t_0, x_0, y) < \eta_0 < 0$.
Define the function $\varphi_\varepsilon : \mathcal{S} \to \mathbb{R}$, for $\varepsilon > 0$ and $z \in \mathcal{S}$, as

$$\varphi_\varepsilon(z) := v_i(z_1) + \partial_t^+ v_i(z_1)(t - t_0) + \partial_x^+ v_i(z_1)(x - x_0) - (\partial_x^+ v_i(z_1) + \psi_i(y_1) + \frac{\eta_0}{2})(y - y_1) - \frac{1}{2\varepsilon}(y - y_1)^2.$$

Since v_i is continuous and non-decreasing function in y, it holds, as established in Rudin [40], that

$$\limsup_{y \to y_1^-} \partial_y^- v_i(t_0, x_0, y) \leq \partial_y^- v_i(t_0, x_0, y_1) \quad \text{and} \quad \partial_y^+ v_i(t_0, x_0, y_1) \leq \liminf_{y \to y_1^+} \partial_y^+ v_i(t_0, x_0, y).$$

Using Taylor's expansion, we have that for $t \geq t_0$, $x \geq x_0$ and $\|z - z_1\|$ small enough,

$$v_i(z) - \varphi_\varepsilon(z) \leq g_i(z)(y - y_1) + \mathrm{o}(\|z - z_1\|),$$

where $g_i : \mathcal{S} \to \mathbb{R}$ are non-positive and defined, for all $z \in \mathcal{S}$, as

$$g_i(z) := \begin{cases} -\liminf_{y \to y_1^-} f_0(t_0, x_0, y) + \frac{\eta_0}{2} & \text{if } y < y_1, \\ \frac{\eta_0}{2} & \text{else.} \end{cases}$$

Since $g_i(z)(y - y_1) < \frac{\eta_0}{2}|y - y_1| < 0$, then φ_ε is a test function in $C^2(\mathcal{S})$ such that z_1 achieves a local maximum of $v_i - \varphi_\varepsilon$ and $v_i(z_1) = \varphi_\varepsilon(z_1)$. Using Theorem 4.2, we get that v_i is a viscosity subsolution of (13). Therefore,

$$\max\left(-\left(\frac{\partial \varphi_\varepsilon}{\partial t} + \mathcal{L}\varphi_\varepsilon + \sum_{j \neq i} Q_{ij}(v_j - \varphi_\varepsilon) \right)(z_1), \ -\left(\frac{\partial \varphi_\varepsilon}{\partial x} + \frac{\partial \varphi_\varepsilon}{\partial y} + \psi_i \right)(z_1) \right) \leq 0.$$

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We know that

$$\mathcal{L}\varphi_\varepsilon(z_1) = -\frac{1}{2\varepsilon}\sigma^2(y_1) - \big(\partial_x^+ v_i(z_1) + \psi_i(y_1) + \frac{\eta_0}{2}\big)\big(h(y_1) + \lambda_t \int_{\mathbb{R}} q(y_1, z)\nu(\mathrm{d}z)\big) - \frac{1}{2\varepsilon}\lambda_t \int_{\mathbb{R}} q^2(y_1, z)\nu(\mathrm{d}z).$$

Hence, we have $\lim_{\varepsilon \to 0} -\mathcal{L}\varphi_\varepsilon(z_1) = +\infty$, while $-\big(\mathcal{L}\varphi_\varepsilon + \sum_{j \neq i} Q_{ij}(v_j - v_i)\big)(z_1) \leq 0$ which leads to a contradiction and the initial assumption does not hold true.

Case 2. $\liminf_{y \to y_1^-} f_0(t_0, x_0, y) = 0$.

Since $\liminf_{y \to y_1^-} f_0(t_0, x_0, y) \leq \limsup_{y \to y_1^-} f_0(t_0, x_0, y) \leq 0$, then $\lim_{y \to y_1^-} f_0(t_0, x_0, y) = f_0(t_0, x_0, y_1)$. Hence, $y \mapsto f_0(t_0, x_0, y)$ is left-continuous in y_1. The fact that $f_0(t_0, x_0, y) < 0$ for almost every $y < y_1$, combined with $f_0(t_0, x_0, y_1) = 0$, and the left-continuity of f_0, implies the existence of $\delta_0 > 0$ such that $y \mapsto f_0(t_0, x_0, y)$ is increasing on $[y_1 - \delta_0, y_1]$. Moreover, since ψ_i is continuous on $\mathbb{R}_+$, then $y \mapsto \partial_y^- v_i(t_0, x_0, y)$ is also continuous on the interval $[y_1 - \delta_0, y_1]$. Given that ψ_i is non-decreasing on $\mathbb{R}_+$ and positive, and that

$$\partial_y^- v_i(t_0, x_0, y) \geq 0, \quad \forall y > 0,$$

we conclude that for f_0 to vanish at z_1, the function $y \mapsto \partial_y^- v_i(t_0, x_0, y)$ must be decreasing on $[y_1 - \delta_0, y_1]$, otherwise f_0 would remain negative in z_1. Therefore, the map $y \mapsto v_i(t_0, x_0, y)$ is strictly concave on $[y_1 - \delta_0, y_1]$, since the existence of a decreasing left derivative implies strict concavity (see Rockafellar [39, Theorem 24.1]). This means that for $y_2 \in]y_1 - \delta_0, y_1[$, there exists $c > 0$ such that, for all $y \in]y_1 - \delta_0, y_1[$,

$$v_i(t_0, x_0, y) < v_i(t_0, x_0, y_2) + c(y - y_2).$$

For any $\varepsilon > 0$, define the function $\widetilde{\varphi}_\varepsilon : \mathcal{S} \to \mathbb{R}$ by

$$\widetilde{\varphi}_\varepsilon(z) := v_i(z_2) + \partial_t^+ v_i(z_1)(t - t_0) + \partial_x^+ v_i(z_2)(x - x_0) + c(y - y_2) - \frac{1}{2\varepsilon}(y - y_2)^2.$$

It follows that $v_i - \widetilde{\varphi}_\varepsilon$ is negative with respect to y in a neighborhood of z_2, excluding z_2 itself. In particular, $\widetilde{\varphi}_\varepsilon \in C^2(\mathcal{S})$ strictly dominates v_i locally around z_2. Repeating the same reasoning as in the previous case thus leads to a contradiction.

Similarly, we have that

$$(t, x, 0) \in \mathcal{E}_i \quad \Rightarrow \quad (t, x', 0) \in \mathcal{E}_i \quad \text{for all } 0 \leq x' \leq x.$$

As a result, the exercise region always lies below the continuation region with respect to the volume variable. That is, for any fixed $(t_0, x_0) \in [0, T] \times [0, \overline{X}]$, it is not possible to have $0 \leq y_1 < y_2 < y_3$ such that $\{(t_0, x_0, y) : y_2 < y < y_3\} \subseteq \mathcal{E}_i$ while simultaneously $\{(t_0, x_0, y) : y_1 < y < y_2\} \subseteq \mathcal{C}_i$. A similar argument establishes that the exercise region is always to the left of the continuation region $\mathcal{C}_i$ in terms of the inventory dimension when $y = 0$. Consequently, the free boundary must be connected. $\qquad\square$

Remark 4.2. *The non-emptiness assumption in Proposition 4.1 is not restrictive, as it merely excludes trivial cases where $\mathcal{S} = \mathcal{E}_i$ or $\mathcal{S} = \mathcal{C}_i$.*

5 Approximation Scheme

We introduce a numerical scheme to solve the PIDE (13), employing a discretization approach that ensures stability and accuracy in approximating the integral-differential operator. We apply a standard finite difference scheme to discretize the differential terms while handling the integral term using a suitable quadrature rule, following the approach of Cont and Voltchkova [20] in the finite activity case.

Discretization. The time domain $[0, T]$ is discretized into N_T steps with $\Delta t := T/N_T$, such that $t_n := T - n\Delta t$, with $n \in \{0, \dots, N_T\}$. To ensure numerical feasibility, we localize y, which is normally in $\mathbb{R}_+$, by restricting it to a bounded domain of size $\overline{Y}$. The state space is discretized into N_X and N_Y steps, respectively. The spatial domain is discretized into $(N_X + 1) \times (N_Y + 1)$ grid points. The spatial step sizes are defined as $\Delta x := \overline{X}/N_X$ and $\Delta y := \overline{Y}/N_Y$, leading to the spatial grid points

$$(x_k, y_l) := ((k-1)\Delta x, (l-1)\Delta y), \quad \forall (k,l) \in \mathbb{I}_{N_X+1} \times \mathbb{I}_{N_Y+1}.$$

A function evaluated at a grid point t_n is denoted with the superscript n. The matrix $v^n := (v_i^n)_{i \in \mathbb{I}_m}$ has entries representing $v_i^n := \left(v_i^n(x_k, y_l)\right)_{(k,l) \in \mathbb{I}_{N_X+1} \times \mathbb{I}_{N_Y+1}}$.

Finite Difference Scheme. The scheme follows a fully implicit-explicit method to ensure stability. The HJBQVI is discretized, for all $i \in \mathbb{I}_m$ and $n \in \{1, \dots, N_T - 1\}$, as

$$\max\left(-\frac{v_i^{n+1} - v_i^n}{\Delta t} - \mathcal{J}v_i^n - \mathcal{I}v_i^{n+1} - \sum_{j \neq i}(v_j^{n+1} - v_i^{n+1})Q_{ij}^{n+1}, \; -\frac{\partial v_i^{n+1}}{\partial x} - \frac{\partial v_i^{n+1}}{\partial y} - \psi_i\right) = 0,$$

with $\mathcal{J}(g) := \mathcal{K}[\frac{\partial g}{\partial y}, \frac{\partial^2 g}{\partial y^2}] = \frac{\sigma^2(y)}{2}\frac{\partial^2 g}{\partial y^2} - h(y)\frac{\partial g}{\partial y}$, and the operators $\mathcal{K}$ and $\mathcal{I}$ are defined in (18). The first-order derivatives are approximated as

$$\frac{\partial v_i^n}{\partial x}(x_k, y_l) \approx \frac{v_i^n(x_{k+1}, y_l) - v_i^n(x_k, y_l)}{\Delta x}, \quad \text{and} \quad \frac{\partial v_i^n}{\partial y}(x_k, y_l) \approx \frac{v_i^n(x_k, y_{l+1}) - v_i^n(x_k, y_l)}{\Delta y},$$

for all $(k,l) \in \{1, \dots, N_X\} \times \{1, \dots, N_Y\}$. The second-order derivative is discretized using a centered finite difference scheme for interior points, while first-order *Neumann conditions* are imposed at the boundaries. In other words, for all $k \in \mathbb{I}_{N_X+1}$,

$$\frac{\partial^2 v_i^n}{\partial y^2}(x_k, y_l) \approx \begin{cases} \frac{v_i^n(x_k, y_3) - 2v_i^n(x_k, y_2) + v_i^n(x_k, y_1)}{\Delta y^2}, & \text{if } l = 1, \\ \frac{v_i^n(x_k, y_{l+1}) - 2v_i^n(x_k, y_l) + v_i^n(x_k, y_{l-1})}{\Delta y^2}, & \text{if } 2 \leq l \leq N_Y, \\ \frac{v_i^n(x_k, y_{N_Y+1}) - 2v_i^n(x_k, y_{N_Y}) + v_i^n(x_k, y_{N_Y-1})}{\Delta y^2}, & \text{if } l = N_Y + 1. \end{cases}$$

The integral term in $\mathcal{I}$ is discretized using a trapezoidal quadrature scheme, such that for all $(k,l) \in \mathbb{I}_{N_X+1} \times \mathbb{I}_{N_Y+1}$, we have

$$\mathcal{I}v_i^n(x_k, y_l) \approx \lambda_{t_n}\sum_{p=1}^{N_Z} w_p\left(v_i^n(x_k, y_l + q(y_l, z_p)) - v_i^n(x_k, y_l)\right),$$

where z_p are quadrature points, and w_p are corresponding quadrature weights such that $\sum_{p=1}^{N_Z} w_p = \int_{\mathbb{R}} \nu(\mathrm{d}z)$. Since $q(y_l, z_p)$ does not necessarily coincide with a grid point, interpolation is necessary. In such cases, the shifted value $y_l + q(y_l, z_p)$ is approximated using linear interpolation between the closest grid points y_j and y_{j+1}, satisfying $y_l + q(y_l, z_p) \in [y_{j(p)}, y_{j(p)+1}]$. The interpolation formula is given by $v_i^n(x_k, y_l + q(y_l, z_p)) \approx v_i^{n,p}(x_k, y_l)$, with

$$v_i^{n,p}(x_k, y_l) := (1 - \alpha_p)v_i^n(x_k, y_{j(p)}) + \alpha_p v_i^n(x_k, y_{j(p)+1}),$$

where $\alpha_p = \frac{y_l + q(y_l, z_p) - y_{j(p)}}{y_{j(p)+1} - y_{j(p)}}$. Substituting this into the discretized integral, we obtain the formulation

$$\mathcal{I}v_i^n(x_k, y_l) \approx \lambda_{t_n}\sum_{p=1}^{N_Z} w_p\left(v_i^{n,p}(x_k, y_l) - v_i^n(x_k, y_l)\right), \quad \forall (k,l) \in \mathbb{I}_{N_X+1} \times \mathbb{I}_{N_Y+1}.$$

Note that we handle the integral part using an implicit time-stepping approach to circumvent the inversion of the dense matrix $\mathcal{I}$. Define F_i^n and G_i^n such that

$$F_i^n(x_k, y_l) := \left(-\frac{v_i^n - v_i^{n-1}}{\Delta t} - \mathcal{J}v_i^{n-1} - \mathcal{I}v_i^n - \sum_{j \neq i}(v_j^n - v_i^n)Q_{ij}^n \right)(x_k, y_l), \quad \forall (k,l) \in \mathbb{I}_{N_X+1} \times \mathbb{I}_{N_Y+1},$$

$$G_i^n(x_k, y_l) := -\frac{v_i^n(x_{k+1}, y_k) - v_i^n(x_k, y_l)}{\Delta x} - \frac{v_i^n(x_k, y_{l+1}) - v_i^n(x_k, y_l)}{\Delta y} - \psi_i(y_l), \quad \forall (k,l) \in \mathbb{I}_{N_X} \times \mathbb{I}_{N_Y}.$$

Since $\max(F_i^n, G_i^n) = 0$, we define the *active region* as

$$\mathcal{E}_i^n = \{(x_k, y_l) \in [0,\overline{X}] \times [0,\overline{Y}] : F_i^n(x_k, y_l) < G_i^n(x_k, y_l)\},$$

and the *passive region* as

$$\mathcal{C}_i^n = \{(x_n, y_n) \in [0,\overline{X}] \times [0,\overline{Y}] : F_i^n(x_n, y_n) \geq G_i^n(x_n, y_n)\}.$$

Rearranging for all grid points in $\mathcal{E}_i^{n+1}$ results in a linear system formulated as

$$\left((I_m + \Delta t Q^{n+1})\right) \otimes I_m)v^{n+1} + \Delta t \lambda_{t_{n+1}} \sum_{p=1}^{N_Z} w_p(I_m \otimes I_m)(v^{n+1,p} - v^{n+1}) = \left((I_m - \Delta t A) \otimes I_m\right)v^n, \quad (22)$$

where I_m is the identity matrix of size $m \times m$. The matrix A is defined as a tridiagonal matrix of size $(N_Y + 1) \times (N_Y + 1)$ with the structure

$$A := \begin{bmatrix} \mathbb{B}_1 & \mathbb{A}_1 & 0 & 0 & \ldots & 0 \\ \mathbb{C}_2 & \mathbb{B}_2 & \mathbb{A}_2 & 0 & \ldots & 0 \\ 0 & \mathbb{C}_3 & \mathbb{B}_3 & \mathbb{A}_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \mathbb{C}_{N_Y} & \mathbb{B}_{N_Y} & \mathbb{A}_{N_Y} \\ 0 & 0 & 0 & 0 & \mathbb{C}_{N_Y+1} & \mathbb{B}_{N_Y+1} \end{bmatrix}.$$

The coefficients $\mathbb{A}_l, \mathbb{B}_l$ and $\mathbb{C}_l$ are defined, for $l = 1$ and $l = N_y + 1$, as

$$\mathbb{A}_1 := \frac{\sigma^2(y_1)}{2\Delta y^2}, \quad \mathbb{B}_1 := -\frac{\sigma^2(y_1)}{\Delta y^2}, \quad \mathbb{B}_{N_Y+1} := -\frac{\sigma^2(y_{N_Y+1})}{\Delta y^2}, \quad \text{and} \quad \mathbb{C}_{N_Y+1} := \frac{\sigma^2(y_{N_Y+1})}{\Delta y^2}.$$

For $l \in \{2, \ldots, N_Y\}$,

$$\mathbb{A}_l := \frac{\sigma^2(y_l)}{2\Delta y^2} - \frac{h(y_l)}{2\Delta y}, \quad \mathbb{B}_l := -\frac{\sigma^2(y_l)}{\Delta y^2} \quad \text{and} \quad \mathbb{C}_l := \frac{\sigma^2(y_l)}{2\Delta y^2} + \frac{h(y_l)}{2\Delta y}.$$

On the hand, if we are in $\mathcal{C}_i^{n+1}$, we have, for all $(k,l) \in \mathbb{I}_{N_X} \times \mathbb{I}_{N_Y}$,

$$-\frac{v_i^{n+1}(x_{k+1}, y_l) - v_i^{n+1}(x_k, y_l)}{\Delta x} - \frac{v_i^{n+1}(x_k, y_{l+1}) - v_i^{n+1}(x_k, y_l)}{\Delta y} - \psi_i^{n+1}(y_l) = 0.$$

The boundary and terminal conditions are enforced explicitly, ensuring that

$$v_i^{N_T}(x_k, y_l) = \Phi_i^n(y_l + \overline{X} - x_k) - \Phi_i^n(y_l), \quad \text{and} \quad v_i^n(x_{N_X+1}, y_l) = 0, \quad \forall (k,l) \in \mathbb{I}_{N_X+1} \times \mathbb{I}_{N_Y+1}.$$

To ensure numerical stability, we impose a Courant-Friedrichs-Lewy (CFL) condition

$$\Delta t \leq C \min\{(\Delta x)^2, (\Delta y)^2\},$$

where C is a positive constant.

We select the time step based on a reference spatial scale proportional to the domain's upper bound. This follows Morton and Mayers [32] and ensures stability in regions with steep gradients. The following algorithm summarizes the time-stepping scheme.

Algorithm 1 Time-Stepping Scheme for PIDE Solution

1: **Initialize:** Set terminal condition $v_i^{N_T}(x, y) = \Phi_i^{N_T}(y + \overline{X} - x) - \Phi_i^{N_T}(y)$.
2: **for** $n = N_T$ to 1 **do**
3:     Find v^{n-1} that solves the linear system (22): $F^n = 0$.
4:     **for** $i = m$ to 1 **do**
5:         Update v_i^{n-1}: $v_i^{n-1}(x, y) \leftarrow \inf_{a \in \{0, \dots, \overline{X} - x\}} v_i^{n-1}(x + a, y + a) + \Phi_i(y + a) - \Phi_i(y)$.
6:         Find the regions $\mathcal{E}_i^{n-1}$ and $\mathcal{C}_i^{n-1}$.
7:     **end for**
8: **end for**
9: **Output:** Approximate solution $v_i^n(x, y)$.

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6 Numerical Results

In this section, we analyze the optimal trading strategy using the numerical scheme detailed in Section 5. We examine various limit order book shapes and illustrate the structure of the optimal exercise and continuation regions for each case. In this numerical study, we define the coefficients in a multiplicative form as

$$h(y) = cy, \quad \sigma(y) = dy, \quad \text{and} \quad q(y, z) = eyz, \quad \forall y, z \in \mathbb{R}_+,$$

where c, d and e are non-negative constants. The controlled *volume effect* process (4) in this example satisfies both Lipschitz continuity and linear growth assumptions. Here, the drift term is proportional to y, ensuring a Geometric Brownian Motion (GBM) structure, the diffusion coefficient scales with the process to maintain a normal-type distribution, and the jump term remains proportional to the current state. We assume that the jumps, which correspond to block trades by external agents, follow an exponential distribution, meaning that the Lévy measure ν takes the form

$$\nu(\mathrm{d}z) = \eta e^{-\eta z} \mathbb{1}_{\{z > 0\}} \mathrm{d}z, \quad \forall z \in \mathbb{R},$$

where $\eta > 0$ controls the decay rate of large jumps. This choice is in line with the existing literature on the subject, notably Pomponio and Abergel [36]. We propose in this case a price impact model in which the trading size influences the price in a concave manner based on the work of Bouchaud, Farmer, and Lillo [11]. The shape functions are given by the power law family with

$$\mathrm{d}F_i(x) = \frac{\kappa}{(x + 1)^{\gamma_i}} \mathrm{d}x, \quad \forall (i, x) \in \mathbb{I}_m \times \mathbb{R}_+.$$

In other words, for all $(i, x, y) \in \mathbb{I}_m \times \mathbb{R}_+^2$, the functions F_i and ψ_i are given by

$$F_i(x) = \begin{cases} \kappa x, & \text{if } \gamma_i = 0, \\ \kappa \log(x + 1), & \text{if } \gamma_i = 1, \\ \frac{\kappa}{1 - \gamma_i}\left[(x + 1)^{1 - \gamma_i} - 1\right], & \text{else.} \end{cases} \quad \text{and} \quad \psi_i(y) = \begin{cases} \frac{y}{\kappa}, & \text{if } \gamma_i = 0, \\ e^{\frac{y}{\kappa}} - 1, & \text{if } \gamma_i = 1, \\ \left[1 + (1 - \gamma_i)\frac{y}{\kappa}\right]^{\frac{1}{1 - \gamma_i}} - 1, & \text{else.} \end{cases}$$

We use the following default parameters for all numerical results unless stated otherwise. These parameters reflect a square root impact.

Parameter	c	d	e	η	κ	γ_0	λ	$\overline{X}$	$\overline{Y}$	T
Value	0.5	0.1	0.2	1.0	0.8	-1	0.5	4	5	4.0

Table 1: Default parameter values.

6.1 Single Regime Case

To isolate the effects of stochastic resilience and liquidity, we first examine the single-regime case, where the Markov chain I remains fixed in a single state. This is equivalent to setting the transition rate matrix Q to zero, making each state absorbing. By doing so, we can analyze the impact of stochastic *volume effect* independently from the influence of changing liquidity regimes.

First, we examine how the value function responds to changes in various model parameters. The plots in Figure 4 illustrate the impact of drift h, jump intensity λ, and volatility σ on execution costs, where a lower volume effect corresponds to higher liquidity and improved execution. In particular, a higher volatility indicate greater market activity. The left hand side of Figure 4 shows that a higher jump intensity leads to less favorable execution opportunities, thereby raising the overall execution cost. Likewise, increased volatility in the volume process reflects a dynamic market, helping to reduce execution costs. Figure 4 also shows that higher resilience, which reflects a higher mean reversion speed, improves execution by accelerating liquidity replenishment.
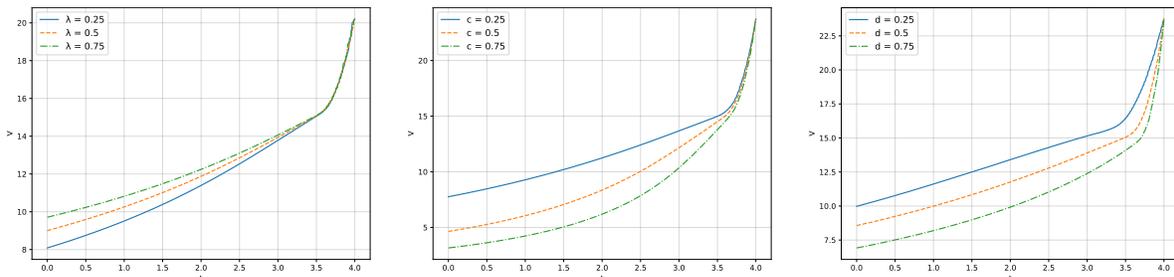
Figure 4: Variation of the value function over time under different market conditions with jump intensity on the left, resilience in the middle, and volatility on the right.

Next, we illustrate the optimal execution strategy under the square root price impact model.
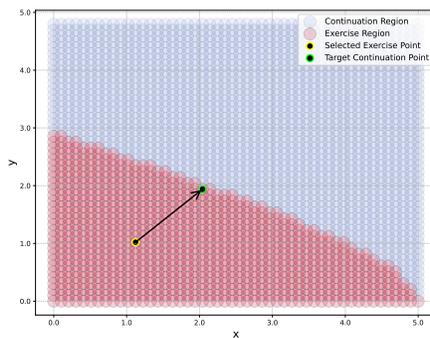
Figure 5: Illustration of continuation (blue) and exercise (red) regions, plotted against purchased quantity (x-axis) and volume effect (y-axis) for $\overline{X} = 5$. The arrow shows an impulse shifting a state from exercise to continuation along $y = x$.

As the x-axis in Figure 5 represents the quantity already acquired, the continuation region expands with increasing x, indicating that larger holdings reduce the urgency to buy more immediately. The curved boundary between the regions reflects the trade-off between acquiring now

or waiting, as price impact increases with trade size but at a decreasing rate. Upon reaching the exercise region, the optimal strategy is to purchase the smallest possible amount to reach the continuation region, thus reducing market impact in an optimal manner.

Subsequently, we verify in Figure 6 the validity of the previous results by examining how variations affect the exercise and continuation regions.
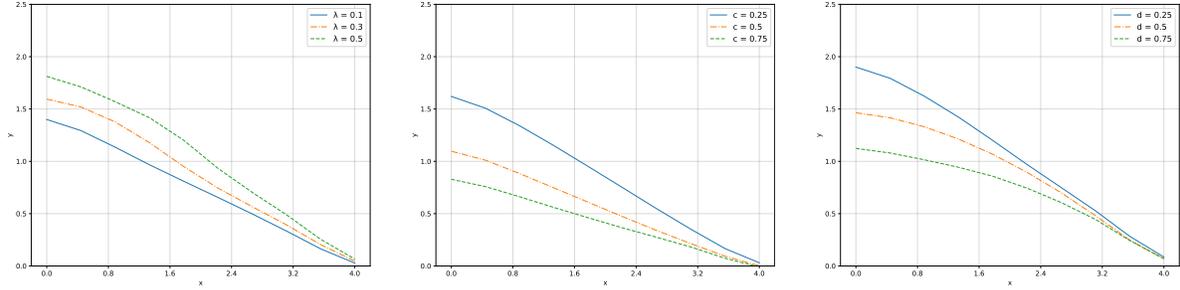

Figure 6: Variation of the exercise boundary $x \mapsto y^*(T/2, x)$ under different market conditions with jump intensity on the left, resilience in the middle, and volatility on the right.

The exercise region expands in response to jumps. Indeed, when jump intensity is high, the optimal strategy involves anticipating increased future price impact by executing larger trades earlier. Conversely, when the volatility of the process governing the volume effect is high, the optimal policy is to acquire fewer shares. This reflects the expectation of more favorable market conditions ahead. The same holds when market resilience increases. This outcome aligns with the previous findings in Figure 4 and those of Alfonsi, Fruth, and Schied [3].

6.2   Regime-Switching case

Next, we extend the analysis to the multi-regime case, where both sources of uncertainty interact. We assume a two-state homogeneous Markov chain, with the transition rate matrix $Q(t)$ at time $t \in [0, T]$ given by

$$Q(t) = \begin{pmatrix} -q_1 & q_1 \\ q_2 & -q_2 \end{pmatrix},$$

where q_1 and q_2 are positive constants. We begin by fixing $q_1 = q_2 = 0.2$ and $\gamma_1 = 0$. With these parameters set, we will then vary the price impact parameter γ_2 to analyze its effect on the continuation and exercise regions of the value functions v_1 and v_2.
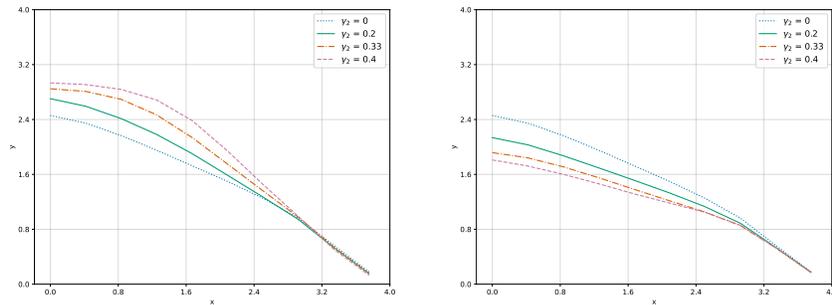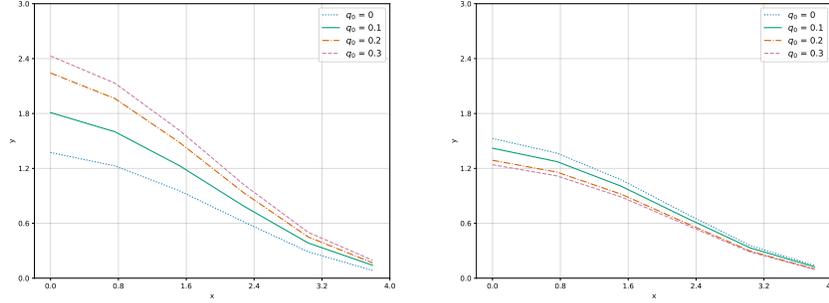

Figure 7: Variation of the exercise boundary $x \mapsto y^*(T/2, x)$ of v_1 on the left and v_2 on the right under different price impacts.

Figure 7 illustrates that when $\gamma_1 < \gamma_2$, the trader anticipates costlier execution in regime 2 and adjusts their strategy accordingly. Indeed, in regime is 1, where the price impact is lower than in regime 2, an increase in the parameter γ_2 leads to an expansion of the exercise region

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(in regime 1), as illustrated on the left-hand side of Figure 7. Conversely, the right-hand side of Figure 7 shows that if we are in regime 2, where the price impact is higher, the exercise region contracts as γ_2 increases. Consequently, the trader increases order sizes in regime 1 to preempt higher execution costs associated with a potential switch to regime 2, while in regime 2, the trader decreases order sizes in anticipation of a favorable transition to regime 1, where trading conditions are more advantageous.

Next, we study the case where the price impact parameter is set to $\gamma_1 = 0$ in regime 1 and $\gamma_2 = 0.33$ in regime 2. We consider $q_1 = q_2 = q_0$ and vary the values of q_0. This setup corresponds to a scenario where there is an equal probability of being in either regime, but the frequency of transitions between regimes changes. The results of this analysis are presented in Figure 8.

Figure 8: Variation of the exercise boundary $x \mapsto y^*(T/2, x)$ of v_1 on the left and v_2 on the right under different regime switching intensities.

A higher switching intensity expands the exercise region in the low price impact regime (see the left-hand side of Figure 8) and contracts it in the high impact regime (see the right-hand side of Figure 8). In regime 1, the increased likelihood of a regime shift prompts the trader to execute larger order sizes in anticipation of deteriorating execution conditions in the former. In the latter (regime 2), the same anticipation leads to smaller order sizes, as the trader expects a transition to a more favorable regime. This behavior is consistent with the earlier observation that traders adjust their execution strategy based on the expected direction of the regime shift and its impact on future trading costs.

Finally, we analyze the case where the price impact parameter is set to $\gamma_1 = 0$ in regime 1 and $\gamma_2 = 0.33$ in regime 2. We fix $q_1 = 0.2$ and vary q_2, allowing us to study how the asymmetry in regime-switching probabilities affects execution.
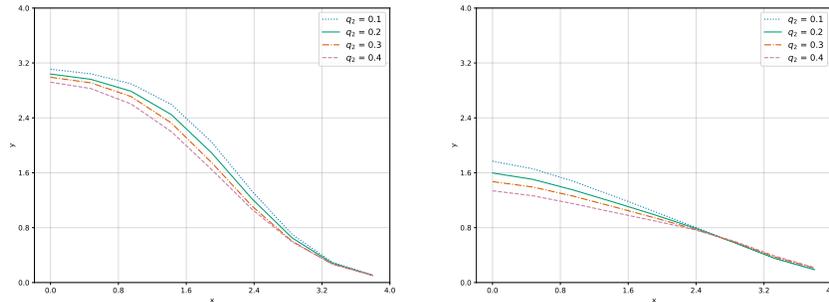

Figure 9: Variation of the exercise boundary $x \mapsto y^*(T/2, x)$ of v_1 on the left and v_2 on the right under different asymmetric regime switching intensities.

Figure 9 confirms our previous findings and highlights the effect of asymmetry in the switching dynamics. The exercise region expands in the low-impact regime when the transition rate matrix implies a higher long-term probability of remaining in the high-impact regime 2. Conversely, as

regime 2 becomes more persistent, the exercise region contracts in regime 2 and expands in regime 1, reflecting adjustments in order sizes based on anticipated execution costs.

References

[1]   Julia Ackermann, Thomas Kruse, and Mikhail Urusov. "Optimal Trade Execution in an Order Book Model with Stochastic Liquidity Parameters". In: SIAM Journal on Financial Mathematics 12.2 (2021), pp. 788–822. DOI: 10.1137/20M135409X.

[2]   Aurélien Alfonsi and Pierre Blanc. "Dynamic optimal execution in a mixed-market-impact Hawkes price model". In: Finance and Stochastics 20 (Jan. 2016). DOI: 10.1007/s00780-015-0282-y.

[3]   Aurélien Alfonsi, Antje Fruth, and Alexander Schied. "Optimal execution strategies in limit order books with general shape functions". In: Quantitative Finance 10.2 (2010), pp. 143–157. DOI: 10.1080/14697680802595700.

[4]   Robert Almgren. "Optimal Trading with Stochastic Liquidity and Volatility". In: SIAM Journal on Financial Mathematics 3.1 (2012), pp. 163–181. DOI: 10.1137/090763470.

[5]   Marco Avellaneda and Sasha Stoikov. "High Frequency Trading in a Limit Order Book". In: Quantitative Finance 8 (Apr. 2008), pp. 217–224. DOI: 10.1080/14697680701381228.

[6]   Peter Bank and Nicole El Karoui. "A stochastic representation theorem with applications to optimization and obstacle problems". In: The Annals of Probability 32.1A (2004), pp. 103–136.

[7]   Erhan Bayraktar and Michael Ludkovski. "Optimal Trade Execution in Illiquid Markets". In: Mathematical Finance 21.4 (2011), pp. 681–701. DOI: https://doi.org/10.1111/j.1467-9965.2010.00446.x.

[8]   Dirk Becherer, Todor Bilarev, and Peter Frentrup. "Optimal liquidation under stochastic liquidity". In: Finance and Stochastics 22.1 (Jan. 2018), pp. 39–68. ISSN: 1432-1122. DOI: 10.1007/s00780-017-0346-2.

[9]   Dimitris Bertsimas and Andrew W. Lo. "Optimal control of execution costs". In: Journal of Financial Markets 1.1 (1998), pp. 1–50. ISSN: 1386-4181. DOI: https://doi.org/10.1016/S1386-4181(97)00012-8.

[10] Bruno Bouchard and Nizar Touzi. "Weak Dynamic Programming Principle for Viscosity Solutions". In: SIAM Journal on Control and Optimization 49.3 (2011), pp. 948–962. DOI: 10.1137/090752328.

[11] Jean-Philippe Bouchaud, J. Doyne Farmer, and Fabrizio Lillo. "How Markets Slowly Digest Changes in Supply and Demand". In: Handbook of Financial Markets: Dynamics and Evolution. North-Holland, 2009, pp. 57–160. DOI: https://doi.org/10.1016/B978-012374258-2.50006-3.

[12] X. Brokmann, E. Sérié, J. Kockelkoren, and J.-P. Bouchaud. "Slow Decay of Impact in Equity Markets". In: Market Microstructure and Liquidity 01.02 (2015), p. 1550007. DOI: 10.1142/S2382626615500070.

[13] Luis A. Caffarelli. "The regularity of free boundaries in higher dimensions". In: Acta Mathematica 139.1 (1977), pp. 155–184. DOI: 10.1007/BF02392236.

[14] Álvaro Cartea and Leandro Sánchez-Betancourt. "Optimal execution with stochastic delay". In: Finance and Stochastics 27.1 (Jan. 2023), pp. 1–47. ISSN: 1432-1122. DOI: 10.1007/s00780-022-00491-w.

[15] Thomas Cayé, Martin Herdegen, and Johannes Muhle-Karbe. "Trading with Small Nonlinear Price Impact". In: The Annals of Applied Probability 30.2 (2020), pp. 706–746. ISSN: 10505164, 21688737.

[16] Etienne Chevalier, Yadh Hafsi, and Vathana Ly Vath. "Optimal Execution under Incomplete Information". In: (2024). DOI: 10.48550/arXiv.2411.04616.

[17] Etienne Chevalier, Yadh Hafsi, and Vathana Ly Vath. "Uncovering Market Disorder and Liquidity Trends Detection". In: (2023). DOI: 10.48550/arXiv.2310.09273.

[18] Chi Chung Siu, Ivan Guo, Song-Ping Zhu, and Robert J. Elliott. "Optimal execution with regime-switching market resilience". In: Journal of Economic Dynamics and Control 101 (2019), pp. 17–40. ISSN: 0165-1889. DOI: https://doi.org/10.1016/j.jedc.2019.01.006.

[19] Katia Colaneri, Zehra Eksi, Rudiger Frey, and Michaela Szolgyenyi. "Optimal liquidation under partial information with price impact". In: Stochastic Processes and their Applications 130.4 (2020), pp. 1913–1946. ISSN: 0304-4149. DOI: https://doi.org/10.1016/j.spa.2019.06.004.

[20] Rama Cont and Ekaterina Voltchkova. "A Finite Difference Scheme for Option Pricing in Jump Diffusion and Exponential Lévy Models". In: SIAM Journal on Numerical Analysis 43.4 (2005), pp. 1596–1626. DOI: 10.1137/S0036142903436186.

[21] Michael Crandall, Hitoshi Ishii, and Pierre-Louis Lions. "User's Guide to Viscosity Solutions of Second Order Artial Differential Equations". In: Bulletin of the American Mathematical Society 27 (July 1992). DOI: 10.1090/S0273-0979-1992-00266-5.

[22] Felix Dammann and Giorgio Ferrari. "Optimal execution with multiplicative price impact and incomplete information on the return". In: Finance and Stochastics 27.3 (July 2023), pp. 713–768. ISSN: 1432-1122. DOI: 10.1007/s00780-023-00508-y.

[23] Donald A. Dawson and Zenghu Li. "Stochastic equations, flows and measure-valued processes". In: The Annals of Probability 40.2 (2012), pp. 813–857. DOI: 10.1214/10-AOP629.

[24] Ibrahim Ekren and Johannes Muhle-Karbe. "Portfolio choice with small temporary and transient price impact". In: Mathematical Finance 29.4 (2019), pp. 1066–1115. DOI: https://doi.org/10.1111/mafi.12204.

[25] Alessio Figalli and Joaquim Serra. "On the fine structure of the free boundary for the classical obstacle problem". In: Inventiones mathematicae 215.1 (2019), pp. 311–366. DOI: 10.1007/s00222-018-0827-8.

[26] Jean-Pierre Fouque, Sebastian Jaimungal, and Yuri F. Saporito. "Optimal Trading with Signals and Stochastic Price Impact". In: SIAM Journal on Financial Mathematics 13.3 (2022), pp. 944–968. DOI: 10.1137/21M1394473.

[27] Antje Fruth, Torsten Schoneborn, and Mikhail Urusov. "Optimal trade execution in order books with stochastic liquidity". In: Mathematical Finance 29.2 (2019), pp. 507–541. DOI: https://doi.org/10.1111/mafi.12180.

[28] G. H. Hardy, J. E. Littlewood, and G. Pólya. "Inequalities". In: (1952).

[29] Ulrich Horst and Xiaonyu Xia. "Multi-dimensional optimal trade execution under stochastic resilience". In: Finance and Stochastics 23.4 (Oct. 2019), pp. 889–923. ISSN: 1432-1122. DOI: 10.1007/s00780-019-00394-3.

[30] R. H. Liu. "Optimal stopping of switching diffusions with state dependent switching rates". In: Stochastics 88.4 (2016), pp. 586–605. DOI: 10.1080/17442508.2015.1110152.

[31] Guiyuan Ma, Chi Chung Siu, Song-Ping Zhu, and Robert J. Elliott. "Optimal portfolio execution problem with stochastic price impact". In: Automatica 112 (2020), p. 108739. ISSN: 0005-1098. DOI: https://doi.org/10.1016/j.automatica.2019.108739.

[32] K. W. Morton and D. F. Mayers. Numerical Solution of Partial Differential Equations: An Introduction. 2nd. Cambridge University Press, 2005. ISBN: 9780521607933.

[33] Johannes Muhle-Karbe, Zexin Wang, and Kevin Webster. "Stochastic Liquidity as a Proxy for Nonlinear Price Impact". In: Operations Research 72.2 (2024), pp. 444–458. DOI: 10.1287/opre.2022.0627.

[34] Anna A. Obizhaeva and Jiang Wang. "Optimal trading strategy and supply/demand dynamics". In: Journal of Financial Markets 16.1 (2013), pp. 1–32. ISSN: 1386-4181. DOI: https://doi.org/10.1016/j.finmar.2012.09.001.

[35] Huyên Pham. "Optimal stopping, free boundary, and American option in a jump-diffusion model". In: Applied Mathematics and Optimization 35.2 (1997), pp. 145–164.

[36] Fabrizio Pomponio and Frédéric Abergel. "Trade-throughs: Empirical Facts and Application to Lead-lag Measures". In: Econophysics of Order-driven Markets: Proceedings of Econophys-Kolkata V. Milano: Springer Milan, 2011, pp. 3–16. DOI: 10.1007/978-88-470-1766-5_1.

[37] Silviu Predoiu, Gennady Shaikhet, and Steven Shreve. "Optimal Execution in a General One-Sided Limit-Order Book". In: SIAM J. Financial Math. 2 (Jan. 2011), pp. 183–212. DOI: 10.1137/10078534X.

[38] P.E. Protter. Stochastic Integration and Differential Equations. Stochastic Modelling and Applied Probability. Springer Berlin Heidelberg, 2005. ISBN: 9783540003137.

[39] R. Tyrrell Rockafellar. "Convex Analysis". In: Princeton Mathematical Series 28 (1970).

[40] Walter Rudin. "Principles of Mathematical Analysis". In: (1976).

[41] Jinghai Shao and Taoran Tian. "From the Optimal Singular Stochastic Control to the Optimal Stopping for Regime-Switching Processes". In: SIAM Journal on Control and Optimization 61.3 (2023), pp. 1631–1650. DOI: 10.1137/22M1506705.

[42] Steven E. Shreve and H. Mete Soner. "Optimal investment and consumption with transaction costs". In: Annals of Applied Probability 4.3 (1994), pp. 609–692.

[43] Eduardo Damian Taranto, Giacomo Bormetti, Jean-Philippe Bouchaud, Fabrizio Lillo, and Bence Toth. "Linear models for the impact of order flow on prices. I. History dependent impact models". In: Quantitative Finance 18.6 (2018), pp. 903–915. DOI: 10.1080/14697688.2017.1395903.

[44] Wei-Xing Zhou. "Universal price impact functions of individual trades in an order-driven market". In: Quantitative Finance 12.8 (2012), pp. 1253–1263. DOI: 10.1080/14697688.2010.504733.

A   Proofs of the Results in Section 3

A.1   Continuity Modulus

Proof of Lemma 3. Let $0 \le t \le s \le T$, $x \in [0, \overline{X}]$, $y \ge 0$ and $X \in \mathcal{A}_t(x)$. Suppose ξ is a nonnegative random variable with finite moments. We have

$$\Delta := \mathbb{E}\Big[\Phi_{I_s}\left(Y_s^{t,y,X} + \xi\right) - \Phi_{I_s}\left(Y_s^{t,y,X}\right)\Big] = \mathbb{E}\Big[\int_0^\xi \psi_{I_s}\left(Y_s^{t,y,X} + \zeta\right)\mathrm{d}\zeta\Big].$$

Note that under Assumption (A1), we have that $\psi_{I_s}(y) \le b^{\frac{1}{\beta}} y^{\frac{1}{\beta}}$, for all $y \ge F_{I_s}(a)$. As $y \mapsto \psi_{I_s}(y)$ is non-decreasing on $\mathbb{R}_+$, we have

$$\Delta := \mathbb{E}\Big[\int_0^\xi \psi_{I_s}\left(Y_s^{t,y,X} + \zeta\right)\mathrm{d}\zeta\Big]$$

$$\le \mathbb{E}\Big[\int_0^\xi b^{\frac{1}{\beta}} \left(F_{I_s}(a) + \zeta\right)^{\frac{1}{\beta}} \mathbb{1}_{\{Y_s^{t,y,X} \le F_{I_s}(a)\}}\mathrm{d}\zeta + \int_0^\xi b^{\frac{1}{\beta}}\left(Y_s^{t,y,X} + \zeta\right)^{\frac{1}{\beta}}\mathbb{1}_{\{Y_s^{t,y,X} > F_{I_s}(a)\}}\mathrm{d}\zeta\Big]$$

$$= \sum_{i=1}^m \frac{\beta}{1+\beta} b^{\frac{1}{\beta}} \mathbb{E}\Big[\mathbb{1}_{\{I_s=i\}}\left(F_i(a)\right)^{\frac{\beta+1}{\beta}}\mathbb{1}_{\{Y_s^{t,y,X} \le F_i(a)\}}\Big(\big(1 + \frac{\xi}{F_i(a)}\big)^{\frac{\beta+1}{\beta}} - 1\Big)\Big]$$

$$+ \sum_{i=1}^m \frac{\beta}{1+\beta} b^{\frac{1}{\beta}} \mathbb{E}\Big[\mathbb{1}_{\{I_s=i\}}\left(Y_s^{t,y,X}\right)^{\frac{\beta+1}{\beta}}\mathbb{1}_{\{Y_s^{t,y,X} > F_i(a)\}}\Big(\big(1 + \frac{\xi}{Y_s^{t,y,X}}\big)^{\frac{\beta+1}{\beta}} - 1\Big)\Big]$$

$$\le \sum_{i=1}^m \frac{\beta}{1+\beta} b^{\frac{1}{\beta}} \mathbb{E}\Big[\Big(F_i(a)^{\frac{\beta+1}{\beta}} + \sup_{0 \le u \le T} |Y_u^{t,y,X}|^{\frac{\beta+1}{\beta}}\Big)\Big(\big(1 + \frac{\xi}{F_i(a)}\big)^{\frac{\beta+1}{\beta}} - 1\Big)\Big].$$

Using Cauchy-Schwarz inequality,

$$\Delta \le \sum_{i=1}^m \frac{\beta}{1+\beta} b^{\frac{1}{\beta}} \mathbb{E}\Big[\Big(F_i(a)^{\frac{\beta+1}{\beta}} + \sup_{0 \le u \le T} |Y_u^{t,y,X}|^{\frac{\beta+1}{\beta}}\Big)^2\Big]^{\frac{1}{2}} \mathbb{E}\Big[\Big(\big(1 + \frac{\xi}{F_i(a)}\big)^{\frac{\beta+1}{\beta}} - 1\Big)^2\Big]^{\frac{1}{2}}.$$

Finally, based on Proposition 2.1, we get that

$$\Delta \le \sum_{i=1}^m C_i^y \mathbb{E}\Big[\Big(\big(1 + \frac{\xi}{F_i(a)}\big)^{\frac{\beta+1}{\beta}} - 1\Big)^2\Big]^{\frac{1}{2}}$$

$$= \rho_y(\xi),$$

where $C_i^y := \frac{2\beta}{1+\beta} b^{\frac{1}{\beta}}\Big(F_i(a)^{\frac{2(\beta+1)}{\beta}} + C_T\big(1 + |y|^{\frac{2(\beta+1)}{\beta}}\big)\Big)^{\frac{1}{2}}$. This completes the proof. $\qquad\square$

B   Proofs of the Results in Section 4

B.1   Viscosity Subsolution Property

Proof of Theorem 4.1. Let $z_0 := (t_0, x_0, y_0) \in \mathcal{S}$ and $i \in \mathbb{I}_m$. Consider a test function $\varphi \in C^{1,2}(\mathcal{S})$ such that z_0 achieves a local maximum of $v_i - \varphi$ and $v_i(z_0) = \varphi(z_0)$. In other words, there exists $r_0 > 0$ such that

$$v_i(z) \le \varphi(z), \quad \forall z \in \overline{B}_{r_0}(z_0).$$

Define, for all $(j, z) \in \mathbb{I}_m \times \mathcal{S}$,

$$\Psi(z, j) = \begin{cases} \varphi(z), & \text{if } j = i, \\ v_j(z), & \text{if } j \ne i. \end{cases}$$

Observe that $v_j(t, x, y) \leq \Psi(z, j)$ holds within the neighborhood $\overline{B}_{r_0}(z_0)$. Define $Z_u^{z_0, X}$ as the tuple $(u, X_u, Y_u^{t_0, y_0, X})$ for all $u \in [0, T]$. Next, define τ_{r_0} as the first exit time of $Z^{z_0, X}$ from $\overline{B}_{r_0}(z_0)$, i.e.,

$$\tau_{r_0} = \inf \left\{ u \geq t_0 : Z_u^{z_0, X} \notin \overline{B}_{r_0}(z_0) \right\}.$$

Similarly, let τ_1^I denote the first jump time of the I after time t_0, given by

$$\tau_1^I = \inf \left\{ u \geq t_0 : I_u \neq i \right\}.$$

Furthermore, we introduce the $\mathcal{F}$-stopping time τ_0 such that $\tau_0 \in \left]t_0, (\tau_1^I \wedge \tau_{r_0}) \wedge T \right[$. Note that $\tau_1^I > t_0$ and that $I_u = i$, for all $u \in [t_0, \tau_1^I[$. Consider an admissible control $X \in \mathcal{A}_{t_0}(x_0)$ in which the agent buys $0 \leq \eta < (\bar{X} - x_0) \wedge r_0$ assets at time t_0 and then remains inactive until time $(t_0 + \varepsilon) \wedge \tau_0$, where $\varepsilon > 0$. Using the DPP introduced in Proposition 4, we have

$$\Psi(z_0, i) = \varphi(z_0)$$
$$\leq \mathbb{E}\Big[\Phi_{I_{(t_0+\varepsilon)\wedge\tau_0}}(y_0 + \eta) - \Phi_{I_{(t_0+\varepsilon)\wedge\tau_0^-}}(y_0) + v((t_0 + \varepsilon) \wedge \tau_0, x_0 + \eta, Y_{(t_0+\varepsilon)\wedge\tau_0}^{t_0, y_0, X}, I_{(t_0+\varepsilon)\wedge\tau_0})\Big]$$
$$\leq \mathbb{E}\Big[\Phi_i(y_0 + \eta) - \Phi_i(y_0) + \Psi((t_0 + \varepsilon) \wedge \tau_0, x_0 + \eta, Y_{(t_0+\varepsilon)\wedge\tau_0}^{t_0, y_0, X}, i)\Big].$$

$$(23)$$

Applying Itô's formula and taking the expectation over the interval $[t_0, (t_0 + \varepsilon) \wedge \tau_0]$, while noting that $\mathbb{E}\big[\Psi(Z_{t_0}^{z_0, X}, I_{t_0})\big] = \mathbb{E}\big[\varphi(Z_{t_0}^{z_0, X})\big]$, we obtain

$$\mathbb{E}\Big[\Psi(Z_{(t_0+\varepsilon)\wedge\tau_0}^{z_0, X}, I_{(t_0+\varepsilon)\wedge\tau_0})\Big] = \varphi(t_0, x_0 + \eta, y_0 + \eta)$$
$$+ \mathbb{E}\Big[\int_{t_0}^{(t_0+\varepsilon)\wedge\tau_0} \big(\frac{\partial\varphi}{\partial t} + \mathcal{L}\varphi\big)(u, x_0 + \eta, Y_u^{t_0, y_0, X})\mathrm{d}u\Big] \quad (24)$$
$$+ \mathbb{E}\Big[\int_{t_0}^{(t_0+\varepsilon)\wedge\tau_0} \sum_{j\neq i} Q_{ij}(u)\big(v_j - \varphi\big)(u, x_0 + \eta, Y_u^{t_0, y_0, X})\mathrm{d}u\Big].$$

In the last equality, the martingale expectation vanishes since $\varphi(Z_u^{z_0, X})$ and $D\varphi(Z_u^{z_0, X})$ are bounded, given that $Z_u^{z_0, X} \in \overline{B}_{r_0}(z_0)$ for all $u \in [t_0, (t_0 + \varepsilon) \wedge \tau_0]$. For all $u \in [t_0, (t_0 + \varepsilon) \wedge \tau_0]$, since $Z_u^{z_0, X}$ remains in $\overline{B}_{r_0}(z_0)$, we obtain that $v_i(Z_u^{z_0, X}) \leq \varphi(Z_u^{z_0, X})$. Combining this with relations (23) and (24) yields

$$\mathbb{E}\Big[\int_{t_0}^{(t_0+\varepsilon)\wedge\tau_0} \big(\frac{\partial\varphi}{\partial t} + \mathcal{L}\varphi\big)(u, x_0 + \eta, Y_{u^-}^{t_0, y_0, X}) + \sum_{j\neq i} Q_{ij}(u)\big(v_j - \varphi\big)(u, x_0 + \eta, Y_{u^-}^{t_0, y_0, X})\mathrm{d}u\Big]$$
$$\geq \varphi(z_0) - \varphi(t_0, x_0 + \eta, y_0 + \eta) - \Phi_i(y_0 + \eta) + \Phi_i(y_0). \quad (25)$$

Assume that the agent decides to do nothing, meaning $\eta = 0$. We know from Theorem 3.1 that v_j is continuous on $\bar{\mathcal{S}}$, for all $j \in \mathbb{I}_m$. Additionally, Q is continuous on $\mathbb{R}_+$. Dividing inequality (25) by ε and letting ε going to 0 and recalling that $\varphi(z_0) = v_i(z_0)$, we get that

$$\big(\frac{\partial\varphi}{\partial t} + \mathcal{L}\varphi + \sum_{j\neq i} Q_{ij}(v_j - v_i)\big)(z_0) \geq 0.$$

Assume now that the agent executes a part $\eta > 0$ of the position at time t_0. By sending ε to 0, we get

$$0 \geq -\varphi(t_0, x_0 + \eta, y_0 + \eta) + \varphi(t_0, x_0, y_0) - \Phi_i(y_0 + \eta) + \Phi_i(y_0).$$

It follows from equation ([5](#)) that

$$\int_0^\eta \frac{\partial \varphi}{\partial x}(t_0, x_0 + v, y_0 + \eta)\mathrm{d}v + \int_0^\eta \frac{\partial \varphi}{\partial y}(t_0, x_0, y_0 + v)\,\mathrm{d}v + \int_{y_0}^{y_0+\eta} \psi_i(\xi)\,\mathrm{d}\xi \geq 0.$$

The function ψ_i is left-continuous and non-decreasing on $\mathbb{R}_+$. Dividing by η and letting $\eta \to 0$, we obtain

$$\left(\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y} + \psi_i\right)(z_0) \geq 0.$$

This proves the required subsolution property

$$\max\left(-\left(\frac{\partial \varphi}{\partial t} + \mathcal{L}\varphi + \sum_{j \neq i} Q_{ij}(v_j - v_i)\right)(z_0),\ -\left(\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y} + \psi_i\right)(z_0)\right) \leq 0.$$

$\square$

B.2   Viscosity Supersolution Property

Proof of Theorem [4.2](#). Suppose that v does not satisfy the supersolution property. Then, there exists $i_0 \in \mathbb{I}_m$, $z_0 := (t_0, x_0, y_0) \in \mathcal{S}$, a test function $\bar{\varphi} \in C^{1,2}(\mathcal{S})$ such that z_0 achieves a local minimum of $v_{i_0} - \bar{\varphi}$ and $v_{i_0}(z_0) = \bar{\varphi}(z_0)$, and $\eta > 0$ such that

$$-\left(\frac{\partial \bar{\varphi}}{\partial t} + \mathcal{L}\bar{\varphi} + \sum_{j \neq i_0}(v_j - v_{i_0})Q_{i_0 j}\right)(z_0) < -\eta, \quad \text{and} \quad -\left(\frac{\partial \bar{\varphi}}{\partial x} + \frac{\partial \bar{\varphi}}{\partial y} + \psi_{i_0}\right)(z_0) < -\eta.$$

Since z_0 achieves a local minimum of $v_{i_0} - \bar{\varphi}$, there exists $r_0 > 0$ such that

$$v_{i_0}(z) \geq \bar{\varphi}(z), \quad \forall z \in \bar{B}_{r_0}(z_0).$$

By Theorem [3.1](#), the value functions $(v_j)_{j \in \mathbb{I}_m}$ are continuous on $\bar{\mathcal{S}}$. Moreover, the test function $\bar{\varphi}$ belongs to $C^{1,2}(\mathcal{S})$, and each ψ_{i_0} is non-decreasing and left-continuous. Consequently, there exists $r_1 > 0$ such that $t_0 + r_1 < T$ and for all $z := (t, x, y) \in \bar{B}_{r_1}(z_0)$,

$$-\left(\frac{\partial \bar{\varphi}}{\partial t} + \mathcal{L}\bar{\varphi} + \sum_{j \neq i_0}(v_j - v_{i_0})Q_{i_0 j}\right)(z) < -\eta, \quad \text{and} \quad -\left(\frac{\partial \bar{\varphi}}{\partial x} + \frac{\partial \bar{\varphi}}{\partial y} + \psi_{i_0}\right)(z) < -\eta, \qquad (26)$$

where $\bar{B}_{r_1}(z_0) := \{z \in \mathcal{S} : \|z - z_0\| \leq r_1\}$. Define, for all $(j, z) \in \mathbb{I}_m \times \mathcal{S}$,

$$\Psi(z, j) = \begin{cases} \bar{\varphi}(z), & \text{if } j = i_0, \\ v_j(z), & \text{if } j \neq i_0. \end{cases}$$

Let X and admissible strategy in $\mathcal{A}_{t_0}(x_0)$ and $Z_u^{z,X}$ represent the tuple $\left(u, X_u, Y_u^{t,y,X}\right)$ for all $(u, z) \in [0, T] \times \mathcal{S}$. Additionally, set the radius ε_0 as $\varepsilon_0 := r_0 \wedge r_1$. Define τ_B as the first exit time of $Z^{z_0,X}$ from $\bar{B}_{\varepsilon_0}(z_0)$, i.e.,

$$\tau_B = \inf\left\{u \geq t_0 : Z_u^{z,X} \notin \bar{B}_{r_0}(z_0) \cap \bar{B}_{r_1}(z_0)\right\}$$

for any admissible control $X \in \mathcal{A}_{t_0}(x_0)$. Let τ_1^I be the first jump time after time t_0 of the Markov chain I such that

$$\tau_1^I = \inf\left\{u \geq t_0 : I_u \neq i_0\right\}.$$

Next, we introduce the $\mathcal{F}$-stopping time τ_1 such that $\tau_1 := \tau_1^I \wedge \tau_B$. Note that $\tau_B < T$, $t_0 < \tau_1^I$ and that $I_u = i_0$, for all $u \in [0, \tau_1^I[$. Applying Itô's formula between t_0 and τ_1^-, we obtain

$$
\begin{aligned}
\mathbb{E}\Big[\Psi(Z_{\tau_1^-}^{z_0,X}, I_{\tau_1^-})\Big] = \bar{\varphi}(z_0) &+ \mathbb{E}\Big[\int_0^{\tau_1^-} \big(\frac{\partial \bar{\varphi}}{\partial x} + \frac{\partial \bar{\varphi}}{\partial y}\big)(Z_{u-}^{z_0,X})\,\mathrm{d}X_u^c\Big] \\
&+ \mathbb{E}\Big[\int_{t_0}^{\tau_1^-} \big(\frac{\partial \bar{\varphi}}{\partial t} + \mathcal{L}\bar{\varphi} + \sum_{j\neq i}(v_j - \bar{\varphi})Q_{i_0 j}\big)(Z_{u-}^{z_0,X})\,\mathrm{d}u\Big] \\
&+ \mathbb{E}\Big[\sum_{t_0 \leq u < \tau_1} \bar{\varphi}(u, X_u, Y_u^{t_0,y_0,X}) - \bar{\varphi}(u, X_{u-}, \check{Y}_{u-}^{t_0,y_0,X})\Big].
\end{aligned}
\tag{27}
$$

In the last equality, the martingale expectation vanishes since $\varphi(Z_u^{z_0,X})$ and $D\varphi(Z_u^{z_0,X})$ are bounded, given that $Z_u^{z_0,X} \in \bar{B}_{\varepsilon_0}(z_0)$ for all $u \in [t_0, \tau_1[$. From (26) the fact that $\Delta X_u := X_u - X_{u-} = Y_u^{t_0,y_0,X} - \check{Y}_{u-}^{t_0,y_0,X}$ for all $t_0 \leq u < \tau_1$, we deduce that

$$
\begin{aligned}
\bar{\varphi}(u, X_u, Y_u^{t_0,y_0,X}) - \bar{\varphi}(u, X_{u-}, \check{Y}_{u-}^{t_0,y_0,X}) &= \int_0^{\Delta X_u} \big(\frac{\partial \bar{\varphi}}{\partial x} + \frac{\partial \bar{\varphi}}{\partial y}\big)(u, X_{u-} + v, \check{Y}_{u-}^{t_0,y_0,X} + v)\,\mathrm{d}v \\
&\geq \eta \Delta X_u - \int_0^{\Delta X_u} \psi_{i_0}(\check{Y}_{u-}^{t_0,y_0,X} + \zeta)\,\mathrm{d}\zeta \\
&= \eta \Delta X_u - \Phi_{i_0}(Y_u^{t_0,y_0,X}) + \Phi_{i_0}(\check{Y}_{u-}^{t_0,y_0,X}).
\end{aligned}
\tag{28}
$$

Note that $\mathbb{E}\Big[\Psi(Z_{\tau_1^-}^{z_0,X}, I_{\tau_1^-})\Big] = \mathbb{E}\Big[\bar{\varphi}(Z_{\tau_1^-}^{z_0,X})\Big]$. Plugging (26) and (28) in equation (27), we obtain

$$
\begin{aligned}
\mathbb{E}\Big[\bar{\varphi}(Z_{\tau_1^-}^{z_0,X})\Big] \geq \bar{\varphi}(z_0) &+ \eta\mathbb{E}\Big[\tau_1 - t_0 + \int_{t_0}^{\tau_1^-}\mathrm{d}X_u\Big] - \mathbb{E}\Big[\int_{t_0}^{\tau_1^-}\psi_{i_0}(\check{Y}_{u-}^{t_0,y_0,X})\mathrm{d}X_u^c\Big] \\
&- \mathbb{E}\Big[\sum_{t_0 \leq u < \tau_1}\Phi_{i_0}(Y_u^{t_0,y_0,X}) - \Phi_{i_0}(\check{Y}_{u-}^{t_0,y_0,X})\Big].
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
v_{i_0}(z_0) = \bar{\varphi}(z_0) \leq \mathbb{E}\Big[&\int_{t_0}^{\tau_1^-}\psi_{i_0}(\check{Y}_{u-}^{t_0,y_0,X})\mathrm{d}X_u^c + \bar{\varphi}(Z_{\tau_1^-}^{z_0,X})\Big] \\
&+ \mathbb{E}\Big[\sum_{t_0 \leq u < \tau_1}\Phi_{i_0}(Y_u^{t_0,y_0,X}) - \Phi_{i_0}(\check{Y}_{u-}^{t_0,y_0,X}) - \eta\big(\int_{t_0}^{\tau_1^-}\mathrm{d}u + \int_{t_0}^{\tau_1^-}\mathrm{d}X_u\big)\Big].
\end{aligned}
\tag{29}
$$

Define the set

$$
\mathcal{D}^{\varepsilon_0} := \Big\{\omega \in \Omega : \tau_B(\omega) < \tau_1^I(\omega) \text{ and } \check{Y}_{\tau_1^-}^{t_0,y_0,X}(\omega) \in [y_0 - \varepsilon_0, y_0 + \varepsilon_0]\Big\},
$$

and the $\mathcal{F}_{\tau_1}$-random variable,

$$
Z_{\tau_1}^{(\gamma)} := \big(\tau_1, \check{Y}_{\tau_1^-}^{t_0,y_0,X} + \gamma\Delta X_{\tau_1}, X_{\tau_1^-} + \gamma\Delta X_{\tau_1}\big).
$$

We know that if $\tau_1 = \tau_B$ and $\check{Y}_{\tau_1^-}^{t_0,y_0,X} \in [y_0 - \varepsilon_0, y_0 + \varepsilon_0]$, then $Z_{\tau_1}^{z_0,X}$ either lies on the boundary $\partial\bar{B}_{\varepsilon_0}(z_0)$ or has exited $\bar{B}_{\varepsilon_0}(z_0)$ at time τ_1. Since $\check{Y}_{\tau_1^-}^{t_0,y_0,X}$ accounts for the jumps of M, only the jumps in the inventory process X can cause $Z^{z_0,X}$ to cross the boundary at τ_1. Consequently, there exists an $\mathcal{F}_{\tau_1}$-measurable random variable $\gamma \in [0, 1]$ such that

$$
\check{Y}_{\tau_1^-}^{t_0,y_0,X} + \gamma\Delta X_{\tau_1} \in \big\{y_0 - \varepsilon_0, y_0 + \varepsilon_0\big\}, \quad \text{and} \quad Z_{\tau_1}^{(\gamma)} \in \bar{B}_{\varepsilon_0}(z_0),
$$

on the set $\mathcal{D}^{\varepsilon_0}$. It follows from the DPP (see Lemma 4), that

$$v_{i_0}(Z_{\tau_1}^{(\gamma)}(\omega)) \le v_{i_0}(Z_{\tau_1^-}^{z_0,X}(\omega)) + \Phi_{i_0}(Y_{\tau_1}^{t_0,y_0,X}(\omega)) - \Phi_{i_0}(\check{Y}_{\tau_1^-}^{t_0,y_0,X}(\omega) + \gamma\Delta X_{\tau_1}(\omega)), \quad \forall \omega \in \mathcal{D}^{\varepsilon_0}.$$

Additionally, given that $Z_{\tau_1}^{(\gamma)} \in \bar{B}_{\varepsilon_0}(z_0)$, it follows that, for all $\omega \in \mathcal{D}^{\varepsilon_0}$,

$$\bar{\varphi}(Z_{\tau_1^-}^{z_0,X}(\omega)) = \bar{\varphi}(Z_{\tau_1}^{(\gamma)}(\omega)) - \int_0^{\gamma(\omega)\Delta X_{\tau_1}(\omega)} \left(\frac{\partial\bar{\varphi}}{\partial x} + \frac{\partial\bar{\varphi}}{\partial y}\right)(t, X_{\tau_1^-}(\omega) + v, \check{Y}_{\tau_1^-}^{t_0,y_0,X}(\omega) + v)\,\mathrm{d}v$$

$$\le \bar{\varphi}(Z_{\tau_1}^{(\gamma)}(\omega)) - \eta\gamma(\omega)\Delta X_{\tau_1}(\omega) + \Phi_{i_0}(\check{Y}_{\tau_1^-}^{t_0,y_0,X}(\omega) + \gamma(\omega)\Delta X_{\tau_1}(\omega)) - \Phi_{i_0}(\check{Y}_{\tau_1^-}^{t_0,y_0,X}(\omega))$$

$$\le v_{i_0}(Z_{\tau_1^-}^{z_0,X}(\omega)) - \eta\gamma(\omega)\Delta X_{\tau_1}(\omega) + \Phi_{i_0}(Y_{\tau_1}^{t_0,y_0,X}(\omega)) - \Phi_{i_0}(\check{Y}_{\tau_1^-}^{t_0,y_0,X}(\omega)).$$

Plugging the last inequality in (29), we obtain

$$v_{i_0}(z_0) \le \mathbb{E}\Big[\sum_{t_0 \le u \le \tau_1} \Phi_{i_0}(Y_u^{t_0,y_0,X}) - \Phi_{i_0}(Y_{u^-}^{t_0,y_0,X}) + \int_{t_0}^{\tau_1} \psi_{i_0}(\check{Y}_{u^-}^{t_0,y_0,X})\mathrm{d}X_u^c + v_{i_0}(Z_{\tau_1}^{z_0,X})\Big]$$

$$- \eta\mathbb{E}\Big[\int_{t_0}^{\tau_1^-}\mathrm{d}u + \int_{t_0}^{\tau_1^-}\mathrm{d}X_u + \gamma\Delta X_{\tau_1}\mathbb{1}_{\{\check{Y}_{\tau_1^-}^{t_0,y_0,X}\in[y_0-\varepsilon_0,y_0+\varepsilon_0]\}}\mathbb{1}_{\{\tau_B<\tau_1^I\}}\Big].$$

From the DPP, it follows that

$$0 \le - \inf_{X\in\mathcal{A}_{t_0}(x_0)} \mathbb{E}\Big[\tau_1 - t_0 + X_{\tau_1^-} - x_0 + \gamma\Delta X_{\tau_1}\mathbb{1}_{\{\check{Y}_{\tau_1^-}^{t_0,y_0,X}\in[y_0-\varepsilon_0,y_0+\varepsilon_0]\}}\mathbb{1}_{\{\tau_B<\tau_1^I\}}\Big].$$

Using the sequential characterization of the supremum, we can find a sequence of admissible controls $X^n = (X^n)_{n\ge0}$ such that

$$\mathbb{E}\Big[\tau_1^n - t_0 + X_{\tau_1^{n-}}^n - x_0 + \gamma\Delta X_{\tau_1^n}^n\mathbb{1}_{\{\check{Y}_{\tau_1^{n-}}^{t_0,y_0,X^n}\in[y_0-\varepsilon,y_0+\varepsilon]\}}\mathbb{1}_{\{\tau_B^n<\tau_1^I\}}\Big] \le \frac{1}{n},$$

where $\tau_B^n := \inf\{t \ge t_0 : (t, X_t^n, Y_t^{t_0,y_0,X^n}) \notin \bar{B}_{\varepsilon_0}(z_0)\}$ and $\tau_1^n := \tau_B^n \wedge \tau_1^I$. As $\tau_1^n - t_0 \ge 0$ and $X_{\tau_1^{n-}}^n - x_0 \ge 0$, we have that

$$\max\Big\{\mathbb{E}[\tau_1^n - t_0],\ \mathbb{E}[X_{\tau_1^{n-}}^n - x_0],\ \mathbb{E}\Big[\gamma\Delta X_{\tau_1^n}^n\mathbb{1}_{\{\check{Y}_{\tau_1^{n-}}^{t_0,y_0,X^n}\in[y_0-\varepsilon,y_0+\varepsilon]\}}\mathbb{1}_{\{\tau_B^n<\tau_1^I\}}\Big]\Big\} \le \frac{1}{n}.$$

Moreover, $\gamma\Delta X_{\tau_1^n}^n = y_0 + \varepsilon_0 - \check{Y}_{\tau_1^{n-}}^{t_0,y_0,X^n} \ge y_0 + \varepsilon_0 - y_0 = \varepsilon_0$. Therefore,

$$\begin{aligned}\frac{1}{n} &\ge \varepsilon_0\mathbb{P}\big(y_0 - \varepsilon_0 \le \check{Y}_{\tau_1^{n-}}^{t_0,y_0,X^n} \le y_0 + \varepsilon_0\ ,\ \tau_B^n < \tau_1^I\big) \\ &\ge \varepsilon_0\Big(\mathbb{P}\big(y_0 - \varepsilon_0 \le \check{Y}_{\tau_1^{n-}}^{t_0,y_0,X^n} \le y_0 + \varepsilon_0\big) + \mathbb{P}\big(\tau_B^n < \tau_1^I\big) - 1\Big).\end{aligned} \tag{30}$$

On the other hand, we have

$$\lim_{n\to+\infty}\mathbb{P}(\tau_1^n = t_0) = 1, \quad\text{and}\quad \lim_{n\to+\infty}\mathbb{P}(X_{\tau_1^{n-}}^n = x_0) = 1.$$

Therefore,

$$\lim_{n\to+\infty}\mathbb{P}(\tau_B^n = t_0) = 1, \quad \lim_{n\to+\infty}\mathbb{P}(\check{Y}_{\tau_1^{n-}}^{t_0,y_0,X^n} = y_0) = 1 \quad\text{and}\quad \lim_{n\to+\infty}\mathbb{P}\big(y_0 - \varepsilon_0 \le \check{Y}_{\tau_1^{n-}}^{t_0,y_0,X^n} \le y_0 + \varepsilon_0\big) = 1.$$

Since $\lim_{n\to+\infty} \mathbb{P}\big(\tau_B^n < \tau_1^I\big) = \mathbb{P}\big(t_0 < \tau_1^I\big) = \exp\big(-\int_{t_0}^{t_0} Q_{ii}(u)\,\mathrm{d}u\big) = 1$, we get from inequality (30) when n goes to $+\infty$ that

$$0 \geq \varepsilon_0\Big(1 + \lim_{n\to\infty} \mathbb{P}\big(\tau_B^n < \tau_1^I\big) - 1\Big) = \varepsilon_0 > 0.$$

This leads to a contradiction. Therefore, we obtain the required viscosity supersolution property. $\qquad\square$