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Systemic Robustness: A Mean-Field Particle System Approach

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ABSTRACT

This paper is concerned with the problem of capital provision in a large particle system modeled by stochastic differential equations involving hitting times, which arises from considerations of systemic risk in a financial network. Motivated by Tang and Tsai, we focus on the number or proportion of surviving entities that never default to measure the systemic robustness. First we show that the mean-field particle system and its limit McKean–Vlasov equation are both well-posed by virtue of the notion of minimal solutions. We then establish a connection between the proportion of surviving entities in the large particle system and the probability of default in the McKean–Vlasov equation as the size of the interacting particle system N tends to infinity. Finally, we study the asymptotic efficiency of capital provision for different drift β, which is linked to the economy regime: The expected number of surviving entities has a uniform upper bound if $\beta < 0$; it is of order $\sqrt{N}$ if $\beta = 0$; and it is of order N if $\beta > 0$, where the effect of capital provision is negligible.

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1 | Introduction

Systemic risk in financial markets has emerged as a major research topic since the 2008 financial crisis, see, for example, Fouque and Langsam (2013). It refers to the risk that small losses and defaults can escalate to cause an event affecting large parts of the financial industry. The existing literature on modeling systemic risk relies on network models (Amini, Cont, and Minca 2016; Amini, Minca, and Sulem 2017; Battiston et al. 2012; Chen, Liu, and Yao 2016; Glasserman and Young 2016; Lorenz, Battiston, and Schweitzer 2009; May and Arinaminpathy 2010), quantitative risk measures (Acharya et al. 2017; Chen, Iyengar, and Moallemi 2013), and particle systems (Carmona, Fouque, and Sun 2015; Cuchiero, Reisinger, and Rigger 2024; Cuchiero, Rigger, and Svaluto-Ferro 2023; Delarue et al. 2015a; 2015b; Delarue, Nadtochiy, and Shkolnikov 2022; Giesecke et al. 2015; Hambly, Ledger, and Sø jmark 2019; Nadtochiy and Shkolnikov 2019).

This paper aims to analyze systemic risk in a large banking system[1] with cross-interactions among its members. We adopt a mean-field particle system approach that is intimately linked to Stefan's problem (and its variants), see Bayraktar et al. (2024), Cuchiero, Reisinger, and Rigger (2024), Cuchiero, Rigger, and Svaluto-Ferro (2023), Hambly, Ledger, and Sø jmark (2019), and Nadtochiy and Shkolnikov (2019).

Consider an interconnected financial system of N homogeneous banks, and let $X_t^{N,1}, \dots, X_t^{N,N}$ be their respective capital levels at time $t \geq 0$. A bank becomes insolvent if its capital drops below some threshold, which is set to be zero without loss of generality. In the absence of defaults, the capital processes $(X_t^{N,i}, t \geq 0)$ for $i = 1, \dots, N$ are supposed to follow the stochastic differential equations (SDEs):

$$dX_t^{N,i} = \beta \, dt + dB_t^i, \quad t \geq 0, \tag{1}$$

where the drift $\beta \in \mathbb{R}$ stands for the economy impact on capital growth rate (see, e.g., Hoover and Giarratani 2020), and $B^1, \dots, B^N$ are independent Brownian motions.

If $k \geq 1$ banks default simultaneously at some time, we assume that the capital of each remaining bank suffers an immediate loss equal to $G(k/N)$, which may cause further defaults. Here $G : [-1, 1] \to \mathbb{R}$ [2] is a continuous and nondecreasing function such that $G(0) = 0$. After the default event, the remaining capital processes continue to follow the SDEs (1) until one of them hits zero, and so on. Rephrasing in mathematical language, the capital levels evolve as follows: for $i = 1, \dots, N$,

$$X_t^{N,i} = Z^{N,i} + \beta t + B_t^i - G(L_t^N) \text{ and}$$
$$L_t^N := \frac{1}{N} \sum_{i=1}^N \mathbb{1}_{\{\tau_i^N \leq t\}}, \quad t \geq 0, \tag{2}$$

where $Z^{N,1}, \dots, Z^{N,N}$ are the initial capital levels, modeled as i.i.d. random variables that are independent of Brownian motions $B^1, \dots, B^N$, and $\tau_i^N := \inf\{t \geq 0 : X_t^{N,i} \leq 0\}$ is the default time of bank i. Here L_t^N denotes the fraction of insolvent banks up to and including time t. We are interested in the evolution of $X^{N,i}$ only until its default time τ_i^N from the practical viewpoint, but we let the process $X^{N,i}$ be defined after τ_i^N by Equation (2) for technical purposes.

Letting $N \to \infty$, we get formally the McKean–Vlasov limit as follows:

$$X_t = Z + \beta t + B_t - G(\Lambda_t) \text{ and } \Lambda_t := \mathbb{P}(\tau \leq t), \quad t \geq 0, \tag{3}$$

where Z is a random variable, distributed according to some probability measure that will be specified in Assumption 3, B is an independent Brownian motion, and $\tau := \inf\{t \geq 0 : X_t \leq 0\}$. It is known that the case $G(x) = \alpha x$ for $\alpha > 0$ provides a probabilistic interpretation of the supercooled Stefan problem and has given rise to a rich body of literature, see Bayraktar et al. (2024), Cuchiero, Reisinger, and Rigger (2024), Cuchiero, Rigger, and Svaluto-Ferro (2023), Delarue et al. (2015a), Delarue et al. (2015b), Delarue, Nadtochiy, and Shkolnikov (2022), and Hambly, Ledger, and Sø jmark (2019).

The system transitions between the following two regimes:

- the *well-behaved regime*, in which the system spends most of its time, and during which the system evolves smoothly and defaults are negligible;
- the *systemic crisis regime*, which occurs rarely, and which is characterized by a large group of banks defaulting in a short period of time.

In our setting, a *systemic failure* occurs at $t_{sys} := \{t \geq 0 : \Delta\Lambda_t := \Lambda_t - \Lambda_{t-} > 0\}$ when a non-negligible fraction of defaults occur in a short period and the system passes abruptly from the well-behaved regime to the systemic crisis regime. Systemic risk has gained significant attention in the wake of the financial crisis of 2007–2008 (see Delarue et al. 2015a; Nadtochiy and Shkolnikov 2019), while we focus on *systemic robustness*, that is, the proportion of banks surviving forever. For $t \geq 0$, we denote by the following:

$$S_t^N := \sum_{i=1}^N \mathbb{1}_{\{\tau_i^N > t\}},$$

the number of banks that survive until time t. Thus, S_∞^N (the number of banks surviving forever) or $1 - L_\infty^N$ (the proportion of banks surviving forever) encodes the system's robustness.

Our first goal is to study the expected surviving proportion $\mathbb{E}[S_\infty^N]/N$, which boils down to showing that under suitable conditions,

$$\lim_{N\to\infty} \frac{\mathbb{E}[S_\infty^N]}{N} = \mathbb{P}(\tau = \infty). \tag{4}$$

For $\beta \leq 0$, the identity (4) follows from a simple observation. As Brownian motion hits each level in finite time with probability 1, we have for all $t \geq 0$,

$$X_t^{N,i} \leq Z^{N,i} + B_t^i \text{ and } X_t \leq Z + B_t,$$

which implies that $\mathbb{1}_{\{\tau_i^N = \infty\}} \leq \mathbb{1}_{\{Z^{N,i} + B_t^i > 0, \forall t \geq 0\}} = 0$ and $\mathbb{1}_{\{\tau = \infty\}} \leq \mathbb{1}_{\{Z + B_t > 0, \forall t \geq 0\}} = 0$ with probability 1. Hence, $\mathbb{E}[S_\infty^N] = 0 = \mathbb{P}(\tau = \infty)$, and Equation (4) holds. The case $\beta > 0$ is less obvious, and will be treated in Section 3. Note that both the particle system (2) and the McKean–Vlasov equation (3) may have more than one solution (see Delarue et al. 2015b, Section 3.1), which renders the statement of our desired result (4) more subtle. Among the solutions to Equations (2) and (3), we need to choose a suitable one. To this end, we adopt the *minimal solution* as in Cuchiero, Rigger, and Svaluto-Ferro (2023), and more details will be provided in Section 2.2.

The next problem is to find a feasible way to compute the minimal solution to the McKean–Vlasov equation (3). The idea is to approximate it by a sequence of regularized problems. Note that

$$\exp\left(-\frac{1}{\varepsilon}\int_0^t (X_s^\varepsilon)^- ds\right) \xrightarrow{\varepsilon\to 0} \mathbb{1}_{\{\tau > t\}} \Longrightarrow$$
$$\mathbb{E}\left[\exp\left(-\frac{1}{\varepsilon}\int_0^t (X_s^\varepsilon)^- ds\right)\right] \xrightarrow{\varepsilon\to 0} 1 - \Lambda_t.$$

For $\varepsilon > 0$, we consider the ε-regularized equation:

$$X_t^\varepsilon = Z + \beta t + B_t - G\left(1 - \mathbb{E}\left[\exp\left(-\frac{1}{\varepsilon}\int_0^t (X_s^\varepsilon)^- ds\right)\right]\right), \quad t \geq 0. \quad (5)$$

We show in Theorem 5 that Equation (5) has a unique solution that converges to the minimal solution to the McKean–Vlasov equation (3).

Finally, we consider how a central planner may stabilize the banking system. In Amini, Minca, and Sulem (2017) and Chen, Liu, and Yao (2016), *equity injections*[3] are applied to weaken the propagation of defaults in a network setting.

Mathematically, their model is in discrete time, and the propagation of defaults must end in a finite number of steps. The planner performs an intervention in the form of equity injection in each step. Here we adopt a particle system approach in continuous time. The intervention in our setting is carried out by *gradual capital provision*. Our goal is to analyze how the gradual capital provision may improve the number/proportion of banks in the long term and study the significance of capital provision as $N \to \infty$. During the preparation of this paper, we learned that Cuchiero, Reisinger, and Rigger (2024) considered an optimal control problem in the same underlying framework. Our paper aims at maximizing the expected number/proportion of banks that never default over all gradual capital provision strategies, while their focus is to find out the minimal capital provision subject to given resources up to some finite time T, and assuming that all the banks have the same gradual capital provision strategy.

Assume that the planner is granted one unit capital per unit of time, and the goal is to divide and allocate it among all surviving banks in order to maximize the expected number/proportion of banks that survive forever. An $\mathbb{F}$-progressively measurable process $\phi \equiv \big((\phi_t^1, \dots, \phi_t^N),\ t \geq 0\big)$ taking values in $[0,1]^N$ is called a strategy if

$$\sum_{i=1}^N \phi_t^i \leq 1, \quad t \geq 0,$$

where $\mathbb{F} \equiv \big(\mathcal{F}_t := \sigma(B_s^1, \dots, B_s^N,\ s \in [0,t]),\ t \geq 0\big)$. Here ϕ_t^i stands for the capital provision rate to bank i at time t, and let Φ_N be the collection of all strategies[4]. So the capital processes subject to the strategy $\phi \in \Phi_N$ evolve as follows: for $i = 1, \dots, N$

$$X_t^{\phi,N,i} = Z^{N,i} + \int_0^t (\beta + \phi_s^i) ds + B_t^i - G(L_t^{\phi,N}) \text{ and}$$

$$L_t^{\phi,N} := \frac{1}{N}\sum_{i=1}^N \mathbb{1}_{\{\tau_i^{\phi,N} \leq t\}}, \quad t \geq 0, \quad (6)$$

with $\tau_i^{\phi,N} := \inf\{t \geq 0 : X_t^{\phi,N,i} \leq 0\}$. Accordingly, the number of banks surviving up to time t is

$$S_t^{\phi,N} := \sum_{i=1}^N \mathbb{1}_{\{\tau_i^{\phi,N} > t\}}.$$

We want to examine the significance of capital provision in the large N limit. To be more precise, we show in Theorem 6 that $\sup_{\phi \in \Phi_N} \mathbb{E}[S_\infty^{\phi,N}]$ has different scaling in N depending on whether $\beta < 0$, $\beta = 0$, or $\beta > 0$.

A word on notations For ease of presentation, we drop the superscript N when the context is clear, that is,

$$Z^i \equiv Z^{N,i},\ X^i \equiv X^{N,i},\ \tau_i \equiv \tau_i^N, \text{ and so forth.}$$

Furthermore, we write when there is no confusion:

$$X^i \equiv X^{\phi,i} \equiv X^{\phi,N,i},\ \tau_i \equiv \tau_i^\phi \equiv \tau_i^{\phi,N},\ L^\phi \equiv L^{\phi,N}, \text{ and so forth.}$$

The rest of the paper is organized as follows. We state the main results in Section 2, and the proofs are given in Sections 3–5.

2 | Preliminaries and Main Results

We provide measure-theoretic background in Section 2.1, and then present the main results in Section 2.2.

2.1 | Preliminaries

Given a generic Polish space E, let $\mathcal{P}(E)$ (resp. $\mathcal{P}_\leq(E)$) be the set of probability (subprobability) measures on E. We set $\mathcal{P} := \mathcal{P}(\mathbb{R}_+)$ and $\mathcal{P}_\leq := \mathcal{P}_\leq(\mathbb{R}_+)$. $\mathcal{C}$ (resp. $\mathcal{D}$) stands for the space of continuous (resp. càdlàg) functions on $[-1,\infty)$. Let $\mathcal{C}$ be endowed with the compact convergence induced by the metric d_u, that is,

$$d_u(f,f') := \sum_{n\geq 1} \frac{1}{2^n}(\|f - f'\|_n \wedge 1) \text{ with}$$

$$\|f - f'\|_n := \sup_{t\in[0,n]} |f_t - f'_t|.$$

As for $\mathcal{D}$, we use the Skorokhod M_1 topology, which turns out to be more convenient for our purpose. Denote by d_m the corresponding metric (see Whitt 2002, Chapter 12, for a complete overview). Moreover, we introduce the set

$$\mathbb{M} := \{\ell \in \mathcal{D} \text{ is nondecreasing:}$$

$$\lim_{t\to\infty} \ell_t =: \ell_\infty \leq 1 \text{ and } \ell_t = 0,\ \forall t \in [-1,0)\}$$

and the Lévy distance d defined by

$$d(\ell,\ell') := \inf\left\{\varepsilon > 0 :\ \ell_{t+\varepsilon} + \varepsilon \geq \ell'_t \geq \ell_{t-\varepsilon} - \varepsilon,\ \forall t \geq 0\right\}, \quad \ell,\ell' \in \mathbb{M}.$$

For each $t \geq 0$, define $d_t : \mathbb{M}\times\mathbb{M} \to \mathbb{R}_+$ by $d_t(\ell,\ell') := d(\ell_{t\wedge}, \ell'_{t\wedge})$, and further $\hat{d}$ by

$$\hat{d}(\ell,\ell') := \int_0^\infty e^{-t}(d_t(\ell,\ell') \wedge 1) dt,$$

where $\ell_{t\wedge} \in \mathbb{M}$ is the function stopped at t, that is, $\ell_{t\wedge s} := \ell_{\min(t,s)}$ for all $s \geq -1$. Lemma 1 summarizes the properties of d and $\hat{d}$, and its proof can be read from Whitt (2002).

Lemma 1. *For every $\ell \in \mathbb{M}$, there exists a unique $\theta_\ell \in \mathcal{P}_\leq$ such that $\ell_t = \theta_\ell((-\infty,t])$ for all $t \geq 0$. For any $(\ell^n)_{n\geq 1} \subset \mathbb{M}$ and $\ell \in \mathbb{M}$, we have* (1) $\Longleftrightarrow$ (2) $\Longrightarrow$ (3) $\Longleftrightarrow$ (4)*, where*

1. $\lim_{n\to\infty} d(\ell^n, \ell) = 0$;
2. θ_{ℓ^n} *converges weakly to* θ_ℓ *in* $\mathcal{P}_{\leq}$;
3. $\lim_{n\to\infty} \hat{d}(\ell^n, \ell) = 0$;
4. $\lim_{n\to\infty} \ell^n_t = \ell_t$ *for all points of continuity* $t \geq 0$ *of* ℓ.

We have $(\mathbb{M}, d)$ *and* $(\mathbb{M}, \hat{d})$*, which are Polish spaces. Moreover,* $(\mathbb{M}, \hat{d})$ *is compact.*

2.2 | Main Results

We start with some terminology. Given independent random variables $Z^{N,1}, \dots, Z^{N,N}$ and independent Brownian motions $B^1, \dots, B^N$, $(X^{N,1}, \dots, X^{N,N}, L^N)$ is said to be a solution to Equation (2) if L^N is $\mathbb{F}$-adapted and takes values in $\mathbb{M}$, and the equations of Equation (2) are satisfied with probability 1. Similarly, for a random variable Z and an independent Brownian motion B, the pair (X, Λ) is said to be a solution to Equation (3) if $\Lambda \in \mathbb{M}$ and the equations of Equation (3) hold. Note that $(X^{N,1}, \dots, X^{N,N})$ (resp. X) is fully determined if L^N (resp. Λ) is fixed. Hence, we also say that L^N (resp. Λ) is a solution to Equation (2) (resp. (3)) for the sake of simplicity.

As previously mentioned, uniqueness of their solution cannot be guaranteed for Equation (2) or (3). Thus, we introduce the minimal solution as follows. A solution $(\underline{X}^{N,1}, \dots, \underline{X}^{N,N}, \underline{L}^N)$ (resp. $(\underline{X}, \underline{\Lambda})$) to Equation (2) (resp. (3)) is called minimal if

$$\underline{L}^N_t \leq L^N_t \quad \left(\text{resp. } \underline{\Lambda}_t \leq \Lambda_t\right), \quad t \geq 0,$$

holds for any solution $(X^{N,1}, \dots, X^{N,N}, L^N)$ (resp. (X, Λ)) to Equation (2) (resp. (3)). The minimal solutions, if it exists, must be unique. Theorem 2 ensures the existence of minimal solutions, which yields the well-posedness of Equations (2) and (3). Its proof is given in Section 3.

Theorem 2. *Both Equations (2) and (3) have a unique minimal solution. If* $\beta > 0$*, we have* $\underline{\Lambda}_\infty < \Lambda_\infty$ *for any solution* (X, Λ) *to Equation (3) that is different from the minimal solution.*

From a financial perspective, all banks will default in finite time when $\beta \leq 0$. If $\beta > 0$, then $1 - \underline{\Lambda}_\infty$ is the largest proportion of banks surviving forever in our setting (3).

Next we turn to the asymptotic proportion of banks that survive forever. As G is continuous, it is uniformly continuous restricted on any compact subset. Let $\alpha : [0,1] \to \mathbb{R}_+$ be the modulus of continuity of G, that is, $|G(z) - G(z')| \leq \alpha(|z - z'|)$ for all $z, z' \in [0,1]$. We also need the following technical assumption on the initial distributions.

Assumption 3. $Z^{N,1}, \dots, Z^{N,N}$ are i.i.d. random variables whose probability distribution $\theta^N \in \mathcal{P}$ is such that $(\theta^N, N \geq 1)$ converges weakly to θ as $N \to \infty$.

For any $b \in \mathbb{R}$, denote by $Q_b : \mathbb{R} \to \mathbb{R}$ the translation, that is, $Q_b(z) := z + b$. Theorem 4 relates the expected surviving proportion $\mathbb{E}[S^N_\infty]/N$ to the minimal solution $\underline{\Lambda}$ to the McKean–Vlasov equation (3). Its proof is given in Section 4.1.

Theorem 4. *Assume that there exists* $(\gamma_N, N \geq 1) \subset \mathbb{R}_+$ *such that*

$$\lim_{N\to\infty} \gamma_N = \lim_{N\to\infty} \frac{1}{N\gamma_N^2} = 0 \text{ and } \theta^N = Q_{\alpha(\gamma_N)}(\theta), \quad N \geq 1,$$

where $Q_b(\theta)$ *is the image measure of* θ *by* Q_b*. Taking the minimal solution* $(X^{N,1}, \dots, X^{N,N}, L^N) = (\underline{X}^{N,1}, \dots, \underline{X}^{N,N}, \underline{L}^N)$*, it holds that*

$$\lim_{N\to\infty} \mathbb{E}\left[\left|\frac{S^N_\infty}{N} - (1 - \underline{\Lambda}_\infty)\right|\right] = 0.$$

Theorem 5 shows that the unique solution to the ε-regularized problem (5) converges to the minimal solution $\underline{\Lambda}$ to Equation (3). Its proof is given in Section 4.2.

Theorem 5. *Let the assumptions in Theorem 4 hold, and assume further that* G *is Lipschitz. For every* $\varepsilon > 0$*, Equation (5) has a unique solution* X^ε *such that* X^ε *has continuous paths and* $\lim_{\varepsilon\to 0} X^\varepsilon_t = \underline{X}_t$ *for all* $t \geq 0$.

Finally, we consider the effect of capital provision in the long term. Similar to Equation (2), the equations (6) do not ensure the uniqueness of their solution in general. Hence, we adopt the minimal solution as follows. $(X^{\phi,N,1}, \dots, X^{\phi,N,N}, L^{\phi,N})$ is said to be a solution to Equation (6) if $L^{\phi,N}$ is $\mathbb{F}$-adapted and takes values in $\mathbb{M}$, and the equations of Equation (6) are satisfied with probability 1. We say that $(\underline{X}^{\phi,N,1}, \dots, \underline{X}^{\phi,N,N}, \underline{L}^{\phi,N})$ is a minimal solution to Equation (6) if

$$\underline{L}^{\phi,N}_t \leq L^{\phi,N}_t, \quad t \geq 0,$$

holds for any solution $(X^{\phi,N,1}, \dots, X^{\phi,N,N}, L^{\phi,N})$ to Equation (6).

Theorem 6. *For all* $N \geq 1$ *and* $\phi \in \Phi_N$*, Equation (6) has a unique minimal solution. Assume further that* $G(x) = \alpha x$ *for some* $\alpha \in \mathbb{R}_+$.

i. *If* $\beta < 0$*, then the number of banks surviving forever is finite, that is,*

$$\sup_{\phi\in\Phi_N} \mathbb{E}[S^{\phi,N}_\infty] < -2/\beta, \quad \text{for all } N \geq 1.$$

Here we emphasize that $S^{\phi,N}_\infty$ *corresponds to any solution to Equation (6).*

ii. *If* $\beta = 0$ *and* θ *is compactly supported, then the capital provision allows to maintain the number of banks surviving forever of order* $\sqrt{N}$*, that is,*

$$0 < \liminf_{N\to\infty} \left(\sup_{\phi\in\Phi_N} \mathbb{E}\left[\frac{S^{\phi,N}_\infty}{\sqrt{N}}\right]\right) \leq \limsup_{N\to\infty} \left(\sup_{\phi\in\Phi_N} \mathbb{E}\left[\frac{S^{\phi,N}_\infty}{\sqrt{N}}\right]\right) < \infty.$$

iii. *If* $\beta > 0$ *and* $\alpha = 0$*, then the effect of capital provision is negligible and the proportion of banks surviving forever remains unchanged, that is,*

$$\lim_{N\to\infty} \left(\sup_{\phi\in\Phi_N} \mathbb{E}\left[\frac{S^{\phi,N}_\infty}{N}\right]\right) = \lim_{N\to\infty} \mathbb{E}\left[\frac{S^{\mathbf{0},N}_\infty}{N}\right] > 0,$$

where $\mathbf{0} \in \Phi_N$ *stands for the strategy* $((\phi_t^1 \equiv 0, \dots, \phi_t^N \equiv 0), t \geq 0)$.

The proof of Theorem 6 is given in Section 5. The theorem indicates that the number of surviving banks scales differently with $\beta < 0$ (negative economy), $\beta = 0$ (neutral economy), and $\beta > 0$ (positive economy). Part (i) implies that the number of surviving banks is of constant order (bounded by $-2/\beta$) in a negative economy, so capital provision as economic intervention is far from efficient. Part (ii) shows that in a neutral economy, the number of surviving banks scales as $\sqrt{N}$, which is of the same order as the "Up the River" model (Aldous 2002; Tang and Tsai 2018). Lower and upper bounds will also be derived in Section 5.2. Part (iii) indicates that in a positive economy, the capital provision does not change the proportion of surviving banks asymptotically. The general case where $\alpha \neq 0$ seems to be challenging, and we conjecture that the effect of capital provision is also negligible. Finally, we remark that, by slightly modifying the proofs in Sections 5.1 and 5.3, (i) and (iii) remain true if we replace the unit capital provision by any finite capital provision $c > 0$:

- if $\beta < 0$,

$$\sup_{\phi} \mathbb{E}[S_\infty^{\phi,N}] < -2c/\beta, \quad \text{for all } N \geq 1;$$

- if $\beta < 0$ and $\alpha = 0$,

$$\lim_{N\to\infty} \left(\sup_{\phi} \mathbb{E}\left[\frac{S_\infty^{\phi,N}}{N}\right]\right) = \lim_{N\to\infty} \mathbb{E}\left[\frac{S_\infty^{\mathbf{0},N}}{N}\right],$$

where the above maximization problems are overall ϕ satisfying

$$\sum_{i=1}^{N} \phi_t^i \leq c, \quad t \geq 0.$$

3 | Proof of Theorem 2

The idea of the proof is similar to Cuchiero, Rigger, and Svaluto-Ferro (2023), which relies on a fixed point argument. Here we provide more details specific to our problem.

Let us introduce the operator $\Gamma : \mathcal{D} \to \mathbb{M}$ as follows. For $\ell \in \mathcal{D}$, let $\Gamma[\ell] \in \mathbb{M}$ be defined by $\Gamma[\ell]_t := \mathbb{P}(\tau^\ell \leq t)$ for all $t \geq 0$, where

$$X_t^\ell := Z + \beta t + B_t - G(\ell_t) \text{ and } \tau^\ell := \inf\left\{t \geq 0 : X_t^\ell \leq 0\right\}.$$

We define similarly $\Gamma_N : \mathcal{S}(\mathcal{D}) \to \mathcal{S}(\mathbb{M})$, where $\mathcal{S}(\mathcal{D})$ (resp. $\mathcal{S}(\mathbb{M})$) denotes the set of $\mathbb{F}$-adapted processes taking values in $\mathcal{D}$ (resp. $\mathbb{M}$). For $L \in \mathcal{S}(\mathcal{D})$, define the following:

$$\Gamma_N[L]_t := \frac{1}{N}\sum_{i=1}^{N} \mathbb{1}_{\{\tau_i^L \leq t\}},$$

where for $1 \leq i \leq N$,

$$X_t^{L,i} := Z^i + \beta t + B_t^i - G(L_t) \text{ and } \tau_i^L := \inf\left\{t \geq 0 : X_t^{L,i} \leq 0\right\}.$$

For any $\ell, \ell' \in \mathcal{D}$ (resp. $L, L' \in \mathcal{S}(\mathcal{D})$), we write $\ell \preceq \ell'$ (resp. $L \preceq L'$) if $\ell_t \leq \ell'_t$ for all $t \geq -1$ (resp. $L_t \leq L'_t$ for all $t \geq -1$ almost surely). It is easy to verify that Γ, Γ_N are monotone with respect to $\preceq$, that is, $\ell \preceq \ell' \Longrightarrow \Gamma[\ell] \preceq \Gamma[\ell']$ (resp. $L \preceq L' \Longrightarrow \Gamma_N[L] \preceq \Gamma_N[L']$). In particular, for any $\ell \in \mathbb{M}$ (resp. $L \in \mathcal{S}(\mathbb{M})$), (X^ℓ, ℓ) (resp. $(X^{L,1}, \dots, X^{L,N}, L)$) is a solution to Equation (3) (resp. (2)) if and only if $\Gamma[\ell] = \ell$ (resp. $\Gamma_N[L] = L$).

Our goal is to prove that Γ (resp. Γ_N) has a fixed point corresponding to the minimal solution. To this end, we need the following lemma.

Lemma 7. *For any* $(\ell^n, n \geq 1) \subset \mathbb{M}$ *and* $\ell \in \mathbb{M}$ *satisfying* $\lim_{n\to\infty} \hat{d}(\ell^n, \ell) = 0$, *we have*

$$\limsup_{n\to\infty} \Gamma[\ell^n]_t \leq \Gamma[\ell]_t, \quad \textit{for all } t \geq 0.$$

Proof. Denote $g^n := G(\ell^n)$ and $g := G(\ell)$ for all $n \geq 1$. By Lemma 1, there exist $\theta^n, \theta \in \mathcal{P}_{\leq}$ such that $\ell_t^n = \theta^n([0,t])$ and $\ell_t = \theta([0,t])$ for all $t \geq 0$. By the Portmanteau theorem, we get the following:

$$\limsup_{n\to\infty} \ell_t^n = \limsup_{n\to\infty} \theta^n([0,t]) \leq \theta([0,t]) = \ell_t,$$

which yields $\Gamma[\ell]_t = \mathbb{P}(\tau^\ell \leq t) \geq \mathbb{P}(C)$, where $C := \{\exists s \in [0,t] : Z + \beta s + B_s \leq h_s\}$ and $h_t := G(\limsup_{n\to\infty} \ell_t^n) = \limsup_{n\to\infty} g_t^n$. Then it suffices to prove the following:

$$\limsup_{n\to\infty} \Gamma[\ell^n]_t = \limsup_{n\to\infty} \mathbb{P}(\tau^{\ell^n} \leq t) \leq \mathbb{P}(C).$$

For each $t \geq 0$, set $h_t^n := \sup_{k\geq n} g_t^k$ so that $g_t^n \leq h_t^n \downarrow h_t$ as $n \to \infty$. Hence,

$$\limsup_{n\to\infty} \mathbb{P}(\tau^{\ell^n} \leq t) \leq \lim_{n\to\infty} \mathbb{P}(\exists s \in [0,t] : Z + \beta s + B_s \leq h_s^n) = \mathbb{P}(A),$$

where

$$A := \{\forall n,\ \exists s \in [0,t] : Z + \beta s + B_s \leq h_s^n\} = \{\forall n \geq m, \\ \exists s \in [0,t] : Z + \beta s + B_s \leq h_s^n\}.$$

Here m can be any positive integer. Note that h^n and h are nondecreasing by definition, but may not be right-continuous. It is enough to show $\mathbb{P}(A \setminus C) = 0$. Introducing an equivalent probability measure $\mathbb{Q}$ defined by the following:

$$\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_t := \exp(-\beta B_t - \beta^2 t/2).$$

Then $\mathbb{P}(A \setminus C) = 0 \Longleftrightarrow \mathbb{Q}(A \setminus C) = 0$, and $(\beta s + B_s)_{0\leq s\leq t}$ is a Brownian motion under $\mathbb{Q}$. Thus, it suffices to prove for the case $\beta = 0$. Define

$$\tau^n := \inf\{s \in [0,t] : Z + B_s \leq h_s^n\}, \quad \forall n \geq 1,$$

where we adopt the convention $\inf \emptyset := \infty$. Clearly, $\tau^n \in [0,t]$ on A. By definition $h^n \downarrow h$ as $n \to \infty$, $n \mapsto \tau^n$ is nondecreasing, which implies the existence of $\tau := \lim_{n\to\infty} \tau^n$. In particular, $\tau \in [0,t]$ also holds on A. In the following, we let the event A occur and distinguish four cases.

i. Suppose $\tau^n < \tau$ for all $n \geq 1$. By definition, there is some $t_n \in [\tau^n, \tau)$ such that $x + B_{t_n} \leq h^n_{t_n} \leq h^n_\tau$. Hence, $x + B_\tau = \lim_{n\to\infty}(x + B_{t_n}) \leq \lim_{n\to\infty} h^n_\tau = h_\tau$. Thus, we get the following:

$$A \cap \{\tau^n < \tau,\ \forall n \geq 1\} \subseteq C. \tag{7}$$

ii. If $\tau^n = \tau = t$ for some n, then we have $x + B_t \leq h^n_t$, and further $x + B_t \leq h_t$. Therefore,

$$A \cap \{\exists n : \tau^n = \tau = t\} \subseteq C. \tag{8}$$

iii. If $\tau^n = \tau < t$ for some n, and τ is a point of discontinuity of h, then we have $x + B_\tau = x + B_{\tau^n} \leq h^n_{\tau^n+} = h^n_{\tau+}$. Define $\hat{h}_s := \lim_{n\to\infty} h^n_{s+}$, then $x + B_\tau \leq \hat{h}_\tau$. Namely,

$$x + B_d \leq \hat{h}_d, \tag{9}$$

at some point $d \in D$, where D is the countable subset of points of discontinuity of h. For all $0 \leq u < s \leq t$, we have $h_s = \lim_{n\to\infty} h^n_s \geq \lim_{n\to\infty} h^n_{u+} = \hat{h}_u$. Now suppose that C does not occur, which implies that $x + B_s > h_s$ for all $s \in [0, t]$ and hence $x + B_s > \hat{h}_d$ for all $d \in D$ and $s \in (d, t]$. As Brownian paths oscillate almost surely infinitely often in every interval, for every $d \in [0, t)$, the set $\{x + B_d \leq \hat{h}_d,\ x + B_s > \hat{h}_d,\ \forall s \in (d, t]\}$ is negligible. We conclude by Equation (9) that

$$\mathbb{P}(A \cap \{\exists n : \tau^n = \tau < t,\ \tau \in D\} \setminus C) \leq \sum_{d\in D} \mathbb{P}(A \cap \{\exists n : \tau^n = \tau = d < t\} \setminus C) = 0. \tag{10}$$

iv. Finally, suppose $\tau^{n^*} = \tau < t$ for some n^*, and τ is a point of continuity of h. For any $\varepsilon > 0$, there exists $\delta > 0$ such that $h_{\tau+\delta} \leq h_\tau + \varepsilon$. Hence, for all $n \geq n^*$ large enough, $\tau^{n^*} = \tau^n = \tau$ and

$$x + B_\tau = x + B_{\tau^n} \leq h^n_{\tau^n+\delta} = h^n_{\tau+\delta} \to h_{\tau+\delta} \leq h_\tau + \varepsilon.$$

As ε is arbitrary, we get $x + B_\tau \leq h_\tau$. Therefore,

$$A \cap \{\exists n : \tau^n = \tau < t,\ \tau \notin D\} \subseteq C. \tag{11}$$

Combining Equations (7), (8), (10), and (11), we conclude that $\mathbb{P}(A \setminus C) = 0$ and $\limsup_{n\to\infty} \mathbb{P}(\tau^{\ell^n} \leq t) \leq \mathbb{P}(\tau^\ell \leq t)$. □

Next we show that $\Gamma : (\mathbb{M}, \hat{d}) \to (\mathbb{M}, d)$ is continuous.

Proposition 8. *For any $(\ell^n, n \geq 1) \subset \mathbb{M}$ and $\ell \in \mathbb{M}$, $\lim_{n\to\infty} d(\Gamma[\ell^n], \Gamma[\ell]) = 0$ holds when $\lim_{n\to\infty} \hat{d}(\ell^n, \ell) = 0$.*

Proof. For simplicity, we write $X^{\ell^n} \equiv X^n, X^\ell \equiv X, \tau^{\ell^n} \equiv \tau^n$, and $\tau^\ell \equiv \tau$. We first show that $\lim_{n\to\infty} \Gamma[\ell^n]_t = \Gamma[\ell]_t$ for all $t \in J$, where $J \subset \mathbb{R}_+$ denotes the collection of the points of continuity of $\Gamma[\ell]$. By Lemma 7, we have $\limsup_{n\to\infty} \Gamma[\ell^n]_t \leq \Gamma[\ell]_t$ for all $t \geq 0$, and it remains to prove $\lim_{n\to\infty} (\Gamma[\ell]_t - \Gamma[\ell^n]_t)^+ = 0$ for $t \in J$. Write

$$(\Gamma[\ell]_t - \Gamma[\ell^n]_t)^+ \leq \mathbb{P}(\tau^n > t, \tau \leq t) = \int_{[0,t]} \mathbb{P}(\tau^n > t | \tau = s) d\Gamma[\ell]_s.$$

As in the proof of Lemma 7, we denote $g^n := G(\ell^n)$ and $g := G(\ell)$. We split the integrand in its continuous and jump part, writing $\Gamma[\ell]^c_s$ for the continuous part, which we estimate in the following.

$$\begin{aligned}
&\int_{[0,t]} \mathbb{P}(\tau^n > t | \tau = s) d\Gamma[\ell]^c_s \\
&= \int_{[0,t]} \mathbb{P}(X^n_u > 0, \forall u \leq t | \tau = s) d\Gamma[\ell]^c_s \\
&\leq \int_{[0,t]} \mathbb{P}\left(X^n_s + \beta(u - s) + B_u - B_s - (g^n_u - g^n_s) > 0,\right. \\
&\qquad \left.\forall s \leq u \leq t | \tau = s\right) d\Gamma[\ell]^c_s \\
&\leq \int_{[0,t]} \mathbb{P}(X^n_s + \beta(u - s) + B_u - B_s > 0, \forall s \leq u \leq t | \tau = s) d\Gamma[\ell]^c_s.
\end{aligned}$$

On the event $\{\tau = s\}$, we have $X_s \leq 0$, so $X^n_s \leq X^n_s - X_s = g_s - g^n_s$. We conclude, using the reflection principle (see, e.g., Karatzas and Shreve 1991, Section 2.8A),

$$\begin{aligned}
&\int_{[0,t]} \mathbb{P}(\tau^n > t | \tau = s) d\Gamma[\ell]^c_s \\
&\leq \int_{[0,t]} \mathbb{P}(\beta(u - s) + B_u - B_s > g^n_s - g_s, \forall s \leq u \leq t | \tau = s) d\Gamma[\ell]^c_s \\
&= \int_{[0,t]} \mathbb{P}\left(\inf_{s\leq u\leq t} (\beta(u - s) + B_u - B_s) > g^n_s - g_s\right) d\Gamma[\ell]^c_s \\
&= \int_{[0,t]} \mathbb{P}\left(\inf_{0\leq u\leq t-s} (\beta u + B_u) > g^n_s - g_s\right) d\Gamma[\ell]^c_s \\
&= \int_{[0,t]} \left(\mathcal{N}\left(\frac{g_s - g^n_s + \beta(t - s)}{\sqrt{t - s}}\right)\right. \\
&\qquad \left. - e^{2\beta(g^n_s - g_s)} \mathcal{N}\left(\frac{g^n_s - g_s + \beta(t - s)}{\sqrt{t - s}}\right)\right) d\Gamma[\ell]^c_s,
\end{aligned}$$

where $\mathcal{N}$ denotes the cumulative distribution function of standard normal. Since $\lim_{n\to\infty} g^n_s = g_s$ holds in a co-countable set, the integrand above converges $\Gamma[\ell]^c-$ almost everywhere to 0. Consequently, the integral above vanishes as $n \to \infty$ by the dominated convergence theorem.

For the integral with respect to the jump part, we get the following:

$$\begin{aligned}
\int_{[0,t]} \mathbb{P}(\tau^n > t | \tau = s) d(\Gamma[\ell]_s - \Gamma[\ell]^c_s) &= \sum_{s\leq t} \mathbb{P}(\tau^n > t | \tau = s) \Delta\Gamma[\ell]_s \\
&= \sum_{s<t} \mathbb{P}(\tau^n > t | \tau = s) \mathbb{P}(\tau = s) \\
&= \sum_{s<t} \mathbb{P}(\tau^n > t, \tau = s),
\end{aligned}$$

where we may take the sum over $s < t$ because $t \in J$. For any $s > 0$, we have by the Portmanteau theorem

$$\liminf_{n\to\infty} \ell^n_{s-} \geq \ell_{s-} \implies \liminf_{n\to\infty} g^n_{s-} \geq g_{s-}.$$

For $s < t$, we have $\mathbb{P}(\tau^n > t, \tau = s) \le \mathbb{P}(\forall \varepsilon \in (0, t-s) : Z + \beta(s+\varepsilon) + B_{s+\varepsilon} \ge g^n_{(s+\varepsilon)-}, \tau = s)$. Taking the lim sup yields, by Fatou's lemma,

$$\limsup_{n\to\infty} \mathbb{P}(\tau^n > t, \tau = s) \le \mathbb{P}\big(\forall \varepsilon \in (0, t-s) : Z + \beta(s+\varepsilon) + B_{s+\varepsilon} \ge g_{(s+\varepsilon)-}, \tau = s\big)$$
$$\le \mathbb{P}(Z + \beta s + B_s \ge g_s, \tau = s)$$
$$\le \mathbb{P}(Z + \beta s + B_s = g_s) = 0.$$

Again by the dominated convergence theorem, we get that the sum converges to zero as $n \to \infty$, which proves $\lim_{n\to\infty} \Gamma[\ell^n]_t = \Gamma[\ell]_t$ for every $t \in J$. Hence, $\lim_{n\to\infty} \hat{d}(\Gamma[\ell^n], \Gamma[\ell]) = 0$.

To show $\lim_{n\to\infty} d(\Gamma[\ell^n], \Gamma[\ell]) = 0$, it remains to show $\lim_{n\to\infty} \Gamma[\ell^n]_\infty = \Gamma[\ell]_\infty$. This is straightforward when $\beta \le 0$, and we assume $\beta > 0$ in the remaining of proof. For each $z \in \mathbb{R}$, denote by $p(z)$ the probability that $z + \beta t + B_t$ never hits $(-\infty, 0]$. We have $p(z) = 1 - e^{-2\beta z^+}$, and further $\lim_{z\to\infty} p(z) = 1$. For each $t \ge 0$, we get the following:

$$\left|\Gamma[\ell^n]_\infty - \Gamma[\ell]_\infty\right| = \left|\mathbb{P}(\tau^n = \infty) - \mathbb{P}(\tau = \infty)\right|$$
$$\le \left|\mathbb{P}(\tau^n > t) - \mathbb{P}(\tau^n = \infty)\right| + \left|\mathbb{P}(\tau^n > t) - \mathbb{P}(\tau > t)\right| + \left|\mathbb{P}(\tau > t) - \mathbb{P}(\tau = \infty)\right|$$
$$= \left|\mathbb{P}(\tau^n > t) - \mathbb{P}(\tau^n = \infty)\right| + \left|\Gamma[\ell^n]_t - \Gamma[\ell]_t\right| + \left|\mathbb{P}(\tau > t) - \mathbb{P}(\tau = \infty)\right|.$$

Let us estimate the first and third terms.

$$\left|\mathbb{P}(\tau^n > t) - \mathbb{P}(\tau^n = \infty)\right| = \mathbb{E}[\mathbb{1}_{\{\tau^n > t\}}] - \mathbb{E}[\mathbb{1}_{\{\tau^n > t\}}\mathbb{1}_{\{\tau^n = \infty\}}]$$
$$= \mathbb{E}[\mathbb{1}_{\{\tau^n > t\}}] - \mathbb{E}\big[\mathbb{P}(\tau^n = \infty | \tau^n > t)\mathbb{1}_{\{\tau^n > t\}}\big].$$

On the event $\{\tau^n > t\}$, we have the following:

$$\{\tau^n = \infty\} = \{X^n_t + \beta(s-t) + (B_s - B_t) - (g^n_s - g^n_t) > 0, \forall s \ge t\}$$
$$\supset \{X^n_t - c + \beta(s-t) + (B_s - B_t) > 0, \forall s \ge t\}, \text{ with}$$
$$c := G(1),$$

which yields by the Markov property $\mathbb{P}(\tau^n = \infty | \tau^n > t) \ge p(X^n_t - c)$ and thus

$$\left|\mathbb{P}(\tau^n > t) - \mathbb{P}(\tau^n = \infty)\right| = \mathbb{E}\big[\mathbb{1}_{\{\tau^n > t\}}(1 - \mathbb{P}(\tau^n = \infty | \tau^n > t))\big]$$
$$\le \mathbb{E}\big[\mathbb{1}_{\{\tau^n > t\}}\big(1 - p(X^n_t - c)\big)\big].$$

Now we couple the process X_t by $Y_t := Z + \beta t + B_t$ for all $t \ge 0$, so $X^n_t \ge Y_t - c$. Let $\sigma := \inf\{t \ge 0 : Y_t \le 0\}$, and we have $\tau^n \le \sigma$. Hence,

$$\left|\mathbb{P}(\tau^n > t) - \mathbb{P}(\tau^n = \infty)\right| \le \mathbb{E}\big[\mathbb{1}_{\{\tau^n > t\}}\big(1 - p(X^n_t - c)\big)\big]$$
$$\le \left|\mathbb{E}\big[\mathbb{1}_{\{\sigma > t\}}(1 - p(Y_t - 2c))\big]\right|.$$

Similarly, we can establish $\left|\mathbb{P}(\tau > t) - \mathbb{P}(\tau = \infty)\right| \le \left|\mathbb{E}[\mathbb{1}_{\{\sigma > t\}} (1 - p(Y_t - 2c))]\right|$, and thus

$$\left|\mathbb{P}(\tau^n = \infty) - \mathbb{P}(\tau = \infty)\right| \le 2\mathbb{E}\big[\mathbb{1}_{\{\sigma > t\}}(1 - p(Y_t - 2c))\big] + \left|\Gamma[\ell^n]_t - \Gamma[\ell]_t\right|.$$

As $\lim_{t\to\infty} Y_t = \infty$ holds almost surely, we get by the dominated convergence theorem,

$$\lim_{t\to\infty} \mathbb{E}\big[\mathbb{1}_{\{\sigma > t\}}(1 - p(Y_t - 2c))\big] = 0.$$

For any $\varepsilon > 0$, there exists t_ε such that $\mathbb{E}\left[\mathbb{1}_{\{\sigma > t\}}(1 - p(Y_t - 2c))\right] \le \varepsilon$ for all $t \ge t_\varepsilon$. Fix an arbitrary $t \in J$ with $t > t_\varepsilon$. Then we have the following:

$$\lim_{n\to\infty} \left|\Gamma[\ell^n]_\infty - \Gamma[\ell]_\infty\right| \le 2\varepsilon + \lim_{n\to\infty} \left|\Gamma[\ell^n]_t - \Gamma[\ell]_t\right| = 2\varepsilon,$$

which yields the desired result. □

Now we prove Theorem 2.

Proof of Theorem 2. i. We start with the particle system (2). For each $n \ge 0$, denote by $\Gamma^{(n)}_N$ the nth iterate of the operator Γ_N. First we show $\underline{L}^N := \Gamma^{(N)}_N[0]$ is the minimal solution to Equation (2), where $0 \in \mathbb{M}$ is the function that is identically equal to zero. The monotonicity of Γ_N, combined with the fact $0 \preceq \Gamma_N[0] = \Gamma^{(1)}_N[0]$, yields $\Gamma^{(1)}_N[0] \preceq \Gamma_N\left[\Gamma^{(1)}_N[0]\right] = \Gamma^{(2)}_N[0]$, and thus by induction $\Gamma^{(n-1)}_N[0] \preceq \Gamma^{(n)}_N[0]$ for all $n \ge 1$. Note that $\underline{\Lambda}^N \in \mathbb{M}$ only takes values in $\{k/N : k = 0, \dots, N\}$. For $0 \le n \le N$, define the stopping times

$$\sigma_n := \inf\left\{t \ge 0 : \Gamma^{(n)}_N[0]_t \ge \frac{n}{N}\right\} \in [0, \infty].$$

We claim that Equation (12) holds for all $1 \le n \le N$:

$$\Gamma^{(n-1)}_N[0]_t = \Gamma^{(n)}_N[0]_t, \quad \text{for all } t < \sigma_n. \tag{12}$$

Clearly, $\Gamma^{(0)}_N$ denotes the identity operator. For $n = 1$, observe that the first jump time σ_1 of $\Gamma^{(0)}_N[0]$ coincides with the first time at which any of the drifted Brownian motions $Z^i + \beta t + B^i_t$ hits zero, and $\Gamma^{(1)}_N[0]$ is equal to zero prior to the time σ_1, regardless of $\sigma_1 < \infty$ and $\sigma_1 = \infty$. This means that $\Gamma^{(0)}_N[0] = 0 = \Gamma^{(1)}_N[0]$ holds for all $t < \sigma_1$. For the inductive step, assume that Equation (12) is true for all positive integers up to n. Applying Γ_N to both sides, we obtain the following:

$$\Gamma^{(n)}_N[0]_t = \Gamma^{(n+1)}_N[0]_t, \quad \text{for all } t < \sigma_n.$$

We are done if $\sigma_n = \infty$, and it suffices to deal with the case $\sigma_n < \infty$. We distinguish two cases. In the first case, we suppose $\Gamma^{(n)}_N[0]_{\sigma_n} > n/N$. Therefore, we have $\Gamma^{(n)}_N[0]_{\sigma_n} \ge (n+1)/N$, and hence $\sigma_n = \sigma_{n+1}$, completing the inductive step. For the second case, we have $\Gamma_N^{(n)}[0]_{\sigma_n} = n/N$, and for $t \in (\sigma_n, \sigma_{n+1})$,

$$\frac{n}{N} = \Gamma_N^{(n)}[0]_{\sigma_n} \leq \Gamma_N^{(n)}[0]_t \leq \Gamma_N^{(n+1)}[0]_t < \frac{n+1}{N},$$

which implies that $\Gamma_N^{(n)}[0]$ and $\Gamma_N^{(n+1)}[0]$ agree on $(0, \sigma_{n+1})$, completing the inductive step. Next we prove that $\underline{L}^N = \Gamma_N^{(N)}[0]$ solves the particle system. For $n \geq 0$, repeatedly applying Γ_N, we get the following:

$$\Gamma_N^{(n)}[0]_t = \Gamma_N^{(n+1)}[0]_t = \cdots = \underline{L}_t^N = \Gamma_N[\underline{L}_N]_t, \quad \text{for all } t < \sigma_n.$$

Choosing $n = N$, we obtain $\underline{L}_t^N = \Gamma_N[\underline{L}_N]_t$ on $[0, \sigma_N)$. If $\sigma_N = \infty$, then $\underline{L}^N = \Gamma_N[\underline{L}_N]$. Otherwise, recall that $\underline{L}^N \preceq \Gamma_N[\underline{L}^N]$. So we see that $1 \leq \underline{L}_{\sigma_N}^N = \Gamma_N[\underline{L}^N]_{\sigma_N} \leq 1$, and thus $\underline{L}^N = \Gamma_N[\underline{L}^N]$. To prove the minimality of $\underline{L}^N$, we pick an arbitrary solution L. By definition, we have $0 \preceq L$, and thus by iteration,

$$\underline{L}^N = \Gamma_N^{(N)}[0] \preceq \Gamma_N^{(N)}[L] = L,$$

which implies that $\underline{L}^N$ is indeed the minimal solution.

ii. Next we turn to the McKean–Vlasov equation (3). Define the sequence $(\Gamma^{(n)}[0],\ n \geq 1) \subset \mathbb{M}$, where $\Gamma^{(n)}$ denotes the nth iterate of Γ. By construction, we see that the sequence is nondecreasing with respect to $\preceq$. We show in the following that $\Gamma^{(n)}[0] \xrightarrow{\hat{d}} \underline{\Lambda} \in \mathbb{M}$ exists, and $\underline{\Lambda}$ is the minimal solution to Equation (3). By definition, we have $0 \preceq \Gamma[0]$. For each $t \geq 0$, the sequence $\left(\Gamma^{(n)}[0]_t,\ n \geq 1\right)$ is nondecreasing and lying in [0,1], which implies that we can define $\tilde{\Lambda}$ to be its pointwise limit

$$\tilde{\Lambda}_t := \lim_{n\to\infty} \Gamma^{(n)}[0]_t, \quad \text{for all } t \geq 0.$$

Clearly, $\tilde{\Lambda}$ is nondecreasing with $0 \leq \tilde{\Lambda}_t \leq 1$, so its modification $\underline{\Lambda}_t := \tilde{\Lambda}_{t+}$ lies in $\mathbb{M}$. This implies that $\Gamma^{(n)}[0] \xrightarrow{\hat{d}} \underline{\Lambda}$, and further by Proposition 8,

$$\Gamma[\underline{\Lambda}] = \Gamma\left[\lim_{n\to\infty} \Gamma^{(n)}[0]\right] = \lim_{n\to\infty} \Gamma^{(n+1)}[0] = \underline{\Lambda}.$$

So $\underline{\Lambda}$ solves Equation (3). Suppose Λ is another solution. By definition, it holds that $0 \preceq \Lambda$, and using the monotonicity of Γ, this leads to $\Gamma[0] \preceq \Gamma[\Lambda] = \Lambda$. A straightforward induction shows that $\Gamma^{(n)}[0] \preceq \Lambda$ for all $n \geq 1$. If t is a point of continuity of $\underline{\Lambda}$, then $\underline{\Lambda}_t = \lim_{n\to\infty} \Gamma_t^{(n)}[0] \leq \Lambda_t$, and further $\underline{\Lambda} \preceq \Lambda$ by the right continuity of $\underline{\Lambda}$. This proves that $\underline{\Lambda}$ is the minimal solution.

iii. For any solution (X, Λ) of Equation (3) that is different to $(\underline{X}, \underline{\Lambda})$, there exist $v > u > 0$ and $b > a > 0$ such that $\Lambda_t > b > a > \underline{\Lambda}_t$ for all $t \in [u, v]$. Recall that $\underline{\tau} := \inf\{t \geq 0 : \underline{X}_t \leq 0\}$ and $\tau := \inf\{t \geq 0 : X_t \leq 0\}$, and we want to show

$$\underline{\Lambda}_\infty < \Lambda_\infty \iff \mathbb{P}(\tau = \infty) < \mathbb{P}(\underline{\tau} = \infty) \iff \mathbb{P}(\underline{\tau} = \infty)$$
$$-\mathbb{P}(\tau = \infty) = \mathbb{P}(\tau < \infty, \underline{\tau} = \infty) > 0.$$

Namely, denote by $Y_t := Z + \beta t + B_t$ for $t \geq 0$, then

$$\{\tau < \infty, \underline{\tau} = \infty\} \supseteq \left\{Y_t > \underline{\Lambda}_t : \forall t \in [0, u]\right\} \cap$$
$$\{Y_t \in [a, b] : \forall t \in [u, v]\} \cap \left\{Y_t > \underline{\Lambda}_t : \forall t \geq v\right\},$$

which yields the desired result by the Markov property. □

4 | Proof of Theorems 4 and 5

4.1 | Proof of Theorem 4

Recall that $\mathcal{C}$ (resp. $\mathbb{M}$) is endowed with the metric d_u (resp. $\hat{d}$). Let $\Xi := \mathcal{C} \times \mathbb{M}$ be endowed with the product topology, and we denote the corresponding metric by $d_u \otimes \hat{d}$. Then Ξ is Polish. Define accordingly the map $\iota : \Xi \to \mathcal{D}$ by $\iota(f, \ell) := f - G(\ell)$. For every $t \geq -1$, denote by λ_t the path functionals on $\mathcal{D}$:

$$\lambda_t(x) := \mathbb{1}_{\{\tau(x) \leq t\}} \text{ with } \tau(x) := \inf\{s \geq 0 : x_s \leq 0\}.$$

Set $\lambda(x) := (\lambda_t(x),\ t \geq -1) \in \mathbb{M}$. We say that x satisfies the *crossing property* if

$$\inf_{0 \leq s \leq h} x_{\tau+s} - x_\tau < 0, \quad \text{for all } h > 0,$$

where we abbreviate $\tau(x)$ by τ. The lemma below summarizes the properties of ι and λ, which is an adaptation from Nadtochiy and Shkolnikov (2019). We provide its proof for completeness.

Lemma 9.

i. The embedding $\iota : (\Xi, d_u \otimes \hat{d}) \to (\mathcal{D}, d_m)$ is continuous.

ii. Let $x \in \iota(\Xi)$ satisfy the crossing property. Then for any sequence $(x^n,\ n \geq 1) \subset \iota(\Xi)$ converging to x under d_m, we have $\lim_{n\to\infty} \lambda_t(x^n) = \lambda_t(x)$ for all $t \geq 0$ in a co-countable set.

iii. Assume that $(\eta^n)_{n\geq 1} \subset \mathcal{P}(\Xi)$ is a convergent sequence with limit η. Define $\nu^n := \iota(\eta^n)$ and $\nu := \iota(\eta)$. If $\nu-$ almost every path satisfies the crossing property, that is,

$$\nu\left(\left\{x \in \mathcal{D} : \inf_{0\leq s\leq h} x_{\tau+s} - x_\tau < 0,\ \forall h > 0\right\}\right) = 1,$$

then

$$\left(\int_{\mathcal{D}} \lambda_t(x) d\nu^n(x),\ t \geq -1\right) =: \langle \nu^n, \lambda \rangle \xrightarrow{\hat{d}} \langle \nu, \lambda \rangle$$
$$:= \left(\int_{\mathcal{D}} \lambda_t(x) d\nu(x),\ t \geq -1\right).$$

Proof. i. Let $(f^n, \ell^n) \to (f, \ell)$ in Ξ and T be any point of continuity of ℓ. Then we have $f^n \to f$ in $C([-1, T])$, and thus in $\mathcal{D}([-1, T])$. Moreover, as $G(\ell^n)$ and $G(\ell)$ are nondecreasing and $\lim_{n\to\infty} G(\ell_t^n) = G(\ell_t)$ for each point of continuity t of $G(\ell)$, Whitt (2002, Corollary 12.5.1) shows that $G(\ell^n) \to G(\ell)$ in $\mathcal{D}([-1, T])$. The conclusion follows from Whitt (2002, Theorem 12.7.3).

ii. Let J be a co-countable set consisting of all points of continuity of x. Recall that Whitt (2002, Theorem 13.4.1) ensures $\lim_{n\to\infty} \inf_{-1\leq s\leq t} x_s^n = \inf_{-1\leq s\leq t} x_s$ for all $t \in J$. Hence, for any $t \in J$ such that $\inf_{-1\leq s\leq t} x_s \neq 0$, we have

$$\lim_{n\to\infty} \lambda_t(x^n) = \lim_{n\to\infty} \mathbb{1}_{\{\inf_{-1\leq s\leq t} x_s^n \leq 0\}} = \mathbb{1}_{\{\inf_{-1\leq s\leq t} x_s \leq 0\}} = \lambda_t(x).$$

Now let $t \in J$ with $\inf_{-1\le s\le t} x_s = 0$. Then $t \ge \tau(x)$ by definition. If $t > \tau(x)$, then the crossing property implies that $\inf_{-1\le s\le t} x_s < 0$, which leads to a contraction. It follows that the only point in J where the convergence may fail is $\tau(x)$, so $J \setminus \{\tau(x)\}$ is still a co-countable set on which we have the desired convergence.

iii. For ease of presentation, denote $\langle \nu^n, \lambda_t\rangle =: \ell_t^n$ and $\ell^n := (\ell_t^n)_{t\ge 0} \in \mathbb{M}$. By the compactness of $\mathbb{M}$, we may assume that $\ell_t^n \to \ell_t$ for some $\ell \in \mathbb{M}$ at every point of continuity t of ℓ. Then it suffices to prove $\ell = \langle \nu, \lambda\rangle$. Let $T > 0$ and $g : [0, T] \to \mathbb{R}$ be bounded and measurable. The dominated convergence theorem implies

$$\lim_{n\to\infty} \int_0^T \ell_t^n g_t dt = \int_0^T \ell_t g_t dt.$$

By the Skorokhod representation theorem (see, e.g., Kallenberg 2002, Theorem 4.30), we can write

$$\int_0^T \ell_t^n g_t dt = \mathbb{E}\left[\int_0^T \lambda_t(Y^n) g_t dt\right],$$

where Y^n converges almost surely in $\mathcal{D}$ to Y with $\mathcal{L}(Y^n) = \nu^n$ and $\mathcal{L}(Y) = \nu$. By (ii), it holds with probability 1 that

$$\lim_{n\to\infty} \int_0^T \lambda_t(Y^n) g_t dt = \int_0^T \lambda_t(Y) g_t dt.$$

Using the dominated convergence theorem again, we get the following:

$$\int_0^T \ell_t g_t dt = \lim_{n\to\infty} \mathbb{E}\left[\int_0^T \lambda_t(Y^n) g_t dt\right] = \mathbb{E}\left[\int_0^T \lambda_t(Y) g_t dt\right]$$
$$= \int_0^T \langle \nu, \lambda_t\rangle g_t dt.$$

It follows from the right-continuity of ℓ that $\ell = \langle \nu, \lambda\rangle$ for all $t \le T$. Since $T > 0$ is arbitrary, the claim follows. ☐

Let $(X^{N,1}, \dots, X^{N,N}, L^N)$ be a solution to Equation (2). We define the empirical measures $(\xi^N, N \ge 1)$ and $(\mu^N, N \ge 1)$ taking values in $\mathcal{P}(\Xi)$ and $\mathcal{P}(\mathcal{D})$ by the following:

$$\xi^N := \frac{1}{N}\sum_{i=1}^N \delta_{(F^{N,i}, L^N)} \text{ and } \mu^N := \frac{1}{N}\sum_{i=1}^N \delta_{X^{N,i}}, \tag{13}$$

where $F^{N,i} := (Z^{N,i} + \beta t + B_t^i,\ t \ge 0)$. By definition, we have $\mu^N = \iota(\xi^N)$ for all $N \ge 1$. We need a few preliminary results on the sequence of random measures $(\xi^N, N \ge 1)$.

Lemma 10. *The sequence $(\xi^N)_{N\ge 1}$ is tight under Assumption 3.*

Proof. The assumption yields the tightness of $(Z^{N,1} + \beta t + B_t^1,\ t \ge 0)$, $N \ge 1$, and thus that of $((Z^{N,1} + \beta t + B_t^1,\ t \ge 0), L^N)$, $N \ge 1$ as $(\mathbb{M}, \hat{d})$ is compact. By Sznitman's theorem (see, e.g., Sznitman 1991, Proposition 2.2), $(\xi^N, N \ge 1)$ is tight. ☐

Proposition 11. *Let ξ be any limit point of $(\xi^N)_{N\ge 1}$ and $\mu := \iota(\xi)$.*

i. For almost every realization ω, if $\mathcal{L}((F, L)) = \xi(\omega)$, then $(F_t - F_0 - \beta t,\ t \ge 0)$ is a Brownian motion with respect to the filtration generated by (F, L) In particular, $(F_t - F_0 - \beta t,\ t \ge 0)$ is independent of F_0.

ii. μ satisfies the crossing property with probability 1.

Proof. i. For any $t > s \ge 0$, $n \in \mathbb{N}$, $0 \le s_1 \le \dots \le s_n \le s$ and continuous and bounded functions $\phi_k : \mathbb{R}^2 \to \mathbb{R}$, denote $\Phi(f, \ell) := \Pi_{k=1}^n \phi_k(f_{s_k}, \ell_{s_k})$ for all $(f, \ell) \in \Xi$. We have

$$\mathbb{E}\left[\left(\int_\Xi (f_t - f_s - \beta(t-s))\Phi(f,\ell) d\xi^N(f,\ell)\right)^2\right]$$
$$= \mathbb{E}\left[\left(\frac{1}{N}\sum_{i=1}^N (B_t^i - B_s^i)\Phi(W^{N,i}, L^N)\right)^2\right]$$
$$= \frac{1}{N}\mathbb{E}\left[\left((B_t^1 - B_s^1)\Phi(W^{N,1}, L^N)\right)^2\right]$$
$$+ \frac{N^2 - N}{N^2}\mathbb{E}\left[(B_t^1 - B_s^1)(B_t^2 - B_s^2)\Phi(W^{N,1}, L^N)\Phi(W^{N,2}, L^N)\right]$$
$$= \frac{1}{N}\mathbb{E}\left[\left((B_t^1 - B_s^1)\Phi(W^{N,1}, L^N)\right)^2\right] \le \frac{\|\Phi\|_\infty (t-s)}{N} \longrightarrow 0,$$

where $W^{N,i} = (W_t^{N,i},\ t \ge 0)$ with $W_t^{N,i} := Z^{N,i} + \beta t + B_t^i$. Thus, for almost every realization ω, if $(F, L) \sim \xi(\omega)$, then $(F_t - F_0 - \beta t,\ t \ge 0)$ is a martingale with respect to the filtration generated by (F, L). Furthermore, repeating the same arguments to the process $((F_t - F_0 - \beta t)^2 - t,\ t \ge 0)$, we obtain that

$$\mathbb{E}\left[\left(\int_\Xi [(f_t - f_0 - \beta t) - (f_s - f_0 - \beta s) - (t-s)]\Phi(f,\ell) d\xi(f,\ell)\right)^2\right] = 0,$$

which implies that $((W_t - W_0 - \beta t)^2 - t,\ t \ge 0)$ is also a martingale with respect to the filtration generated by (W, L). We conclude by Lévy's characterization (see, e.g., Kallenberg 2002, Theorem 18.3).

ii. For simplicity, we assume that $\mathcal{L}(\mu^N)$ converges to $\mathcal{L}(\mu)$. Note that $\mathcal{L}(\mu) = \mathcal{L}(\iota(\xi))$. For any fixed $h > 0$, we have the following:

$$\mathbb{E}\left[\mu\left(\left\{x \in \mathcal{D} : \inf_{0\le s\le h} x_{\tau+s} - x_\tau = 0\right\}\right)\right]$$
$$= \mathbb{E}\left[\xi\left(\left\{(f,\ell) \in \Xi : \inf_{0\le s\le h} f_{\tau+s} - f_\tau - (G(\ell_{\tau+s}) - G(\ell_\tau)) = 0\right\}\right)\right]$$
$$\le \mathbb{E}\left[\xi\left(\left\{(f,\ell) \in \Xi : \inf_{0\le s\le h} f_{\tau+s} - f_\tau = 0\right\}\right)\right].$$

As for almost every realization ω, if $\mathcal{L}((F, L)) = \xi(\omega)$, then $\tau = \tau(F - G(L))$ is a stopping time with respect to the filtration generated by (F, L). Since $F - F_0$ is a Brownian motion with respect to the same filtration, the strong Markov property yields the following:

$$\mathbb{E}\left[\xi\left(\left\{(f,\ell) \in \Xi : \inf_{0\le s\le h} f_{\tau_0+s} - f_{\tau_0} = 0\right\}\right)\right] = \mathbb{P}\left(\inf_{0\le s\le h} F_s = 0\right) = 0.$$

Hence,

$$\mu\left(\left\{x \in \mathcal{D} : \inf_{0\le s\le h} x_{\tau_0+s} - x_{\tau_0} = 0\right\}\right) = 0,$$

holds almost surely. Repeating this reasoning for $h = h_n$ for a sequence $(h_n, n \ge 1) \subset \mathbb{R}_+$ such that $\lim_{n\to\infty} h_n = 0$ yields the result. □

Following the arguments in Delarue et al. (2015b), we have the following proposition, which establishes the convergence of random measures (see also Cuchiero, Rigger, and Svaluto-Ferro 2023; Nadtochiy and Shkolnikov 2019).

Proposition 12. *The sequence $(\xi^N, N \ge 1)$ is tight under Assumption 3. Moreover, let ξ be any limit point. Then for almost every realization ω, $\xi(\omega)$ coincides with the law of (F, Λ), where $(F - G(\Lambda), \Lambda)$ is a solution to Equation (3).*

Proof. The tightness of $(\xi^N, N \ge 1)$ follows from Lemma 10. Without loss of generality, we may assume the convergence of $(\xi^N, N \ge 1)$ for ease of presentation. Then we get $\mu^N \to \mu := \iota(\xi)$ by the continuity of ι. Consider the maps

$$t \mapsto \mathbb{E}[\langle \mu, \lambda_t\rangle] \text{ and } t \mapsto \mathbb{E}\left[\int_\Xi \ell_t d\xi(f,\ell)\right],$$

that are nondecreasing and thus have at most countably discontinuities. Denote by J the co-countable set of their points of continuity. Combining Lemma 9 (iii) and Proposition 11 (iii), we deduce that $\langle \mu^N, \lambda\rangle \to \langle \mu, \lambda\rangle$ in $\mathbb{M}$ almost surely. For any $t \in J$, we have $\mathbb{E}[\langle \mu, \lambda_t\rangle] = \mathbb{E}[\langle \mu, \lambda_{t-}\rangle]$, and thus $\langle \mu, \lambda_t\rangle = \langle \mu, \lambda_{t-}\rangle$ almost surely. Therefore, $\langle \mu^N, \lambda\rangle \to \langle \mu, \lambda\rangle$ in $\mathbb{M}$, which yields as follows:

$$\lim_{N\to\infty} \langle \mu^N, \lambda_t\rangle = \langle \mu, \lambda_t\rangle. \tag{14}$$

Let $\pi_0 : \Xi \to \mathbb{R}$ be the projection defined by $\pi(f,\ell) = f_0$. By definition, π_0 is continuous, and hence

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N \delta_{Z^{N,i}} = \lim_{N\to\infty} \pi_0(\xi^N) = \pi_0(\xi),$$

which implies that for almost every realization ω, $\mathcal{L}(\pi_0(\xi(\omega))) = \lim_{N\to\infty} \theta^N = \theta$. By Lemma 11 (i), for almost every realization ω, if $\mathcal{L}(F, L) = \xi(\omega)$, then $(F_t - \beta t, t \ge 0)$ is Brownian motion with $\mathcal{L}(F_0) = \theta$. Hence, it remains to prove the following:

$$\Lambda_t = \mathbb{P}(\tau(F - G(L)) \le t) = \langle \mu(\omega), \lambda_t\rangle, \quad \text{for all } t \ge 0.$$

To this end, note for every $t \in J$,

$$\mathbb{E}\left[\int_\Xi \Big| |\ell_t - \langle \mu, \lambda_t\rangle| - |\ell_t - \langle \mu^N, \lambda_t\rangle| \Big| d\xi^N(f,\ell)\right]$$
$$\le \mathbb{E}\left[|\langle \mu, \lambda_t\rangle - \langle \mu^N, \lambda_t\rangle|\right]$$

which vanishes as $N \to \infty$ by Equation (14) and the dominated convergence theorem. As $t \in J$, we have $\ell_{t-} = \ell_t$ for $\xi-$ almost every ℓ, and thus the map $\mathbb{M} \ni \ell \to \ell_t \in \mathbb{R}_+$ is continuous $\xi-$almost surely. It follows that

$$\mathbb{E}\left[\int_\Xi |\ell_t - \langle \mu, \lambda_t\rangle| d\xi(f,\ell)\right] = \lim_{N\to\infty} \mathbb{E}\left[\int_\Xi |\ell_t - \langle \mu, \lambda_t\rangle| d\xi^N(f,\ell)\right]$$
$$\le \lim_{N\to\infty} \mathbb{E}\left[\int_\Xi |\ell_t - \langle \mu^N, \lambda_t\rangle| d\xi^N(f,\ell)\right] = 0,$$

where the last equality holds since $\langle \mu^N, \lambda\rangle = \ell$ holds for ξ^N-almost every $\ell \in \mathbb{M}$, almost surely. The conclusion follows by letting t range over a countable dense subset of J and using the right-continuity of ℓ. □

Now we prove Theorem 4.

Proof of Theorem 4. First, we show that any limit point ξ is equal to $\mathcal{L}(\underline{X}, \underline{\Lambda})$ almost surely. Consider the perturbed system: for $i = 1, \dots, N$,

$$\tilde{X}_t^{L,i} = \tilde{Z}^{N,i} + \beta t + B_t^i - G(L_t), \quad \text{for all } t \ge 0$$

and

$$\tilde{\Gamma}_N[L]_t := \frac{1}{N}\sum_{i=1}^N \mathbb{1}_{\{\tilde{\tau}_i^L \le t\}} \text{ and } \tilde{\tau}_i^L := \inf\{t \ge 0 : \tilde{X}_t^{L,i} \le 0\},$$

where $\mathcal{L}(\tilde{Z}^{N,i}) = \theta$ for $1 \le i \le N$. Taking $L \equiv \underline{\Lambda}$, we have the following:

$$\tilde{X}_t^{\underline{\Lambda},i} = \tilde{Z}^{N,i} + \beta t + B_t^i - G(\underline{\Lambda}_t), \quad \text{for all } t \ge 0,$$

and $\tilde{\tau}_i^{\underline{\Lambda}}$ are i.i.d. random variables that are identical in law to $\underline{\tau} := \inf\{t \ge 0 : \underline{X}_t \le 0\}$. It follows from the inequality of Dvoretzky–Kiefer–Wolfowitz (Massart 1990) that

$$\mathbb{P}\left(\|\tilde{\Gamma}_N[\underline{\Lambda}] - \underline{\Lambda}\|_\infty > \gamma_N\right) \le 2e^{-2N\gamma_N^2}, \quad \text{for all } N \ge 1.$$

For each $N \ge 1$, define $A_N := \{\|\tilde{\Gamma}_N[\underline{\Lambda}] - \underline{\Lambda}\|_\infty \le \gamma_N\}$ and the operator $\hat{\Gamma}_N$ by

$$\hat{\Gamma}_N[L]_t := \frac{1}{N}\sum_{i=1}^N \mathbb{1}_{\{\hat{\tau}_i^L \le t\}},$$

where for $i = 1, \dots, N$,

$$\hat{X}_t^{L,i} = \tilde{Z}^{N,i} + \alpha(\gamma_N) + \beta t + B_t^i - G(L_t) \text{ and } \hat{\tau}_i^L = \inf\left\{t \ge 0 : \hat{X}_t^{L,i} \le 0\right\}.$$

In particular, $\mathcal{L}(\tilde{Z}^{N,i} + \alpha(\gamma_N)) = \theta^N = \mathcal{L}(Z^{N,i})$ for $1 \le i \le N$. On the set A_N, we have

$$\underline{\Lambda}_t \ge \tilde{\Gamma}_N[\underline{\Lambda}]_t - \gamma_N, \quad \text{for all } t \ge 0.$$

Hence, $\hat{\Gamma}_N[0] - \gamma_N \preceq \tilde{\Gamma}_N[-\gamma_N] - \gamma_N \preceq \underline{\Lambda}$ holds as $-G(-\gamma_N) - \alpha(\gamma_N) \le -G(0) = 0$. Using the monotonicity again, we get the following:

$$\hat{\Gamma}_N^{(2)}[0] - \gamma_N = \hat{\Gamma}_N[\hat{\Gamma}_N[0]] - \gamma_N \preceq \tilde{\Gamma}_N[\hat{\Gamma}_N[0] - \gamma_N]$$
$$-\gamma_N \preceq \tilde{\Gamma}_N[\underline{\Lambda}] - \gamma_N \preceq \underline{\Lambda},$$

where the second inequality still follows from the uniform continuity of G, that is,

$$-G\big(\hat{\Gamma}_N[0]_t-\gamma_N\big)\le\alpha(\gamma_N)-G\big(\hat{\Gamma}_N[0]_t\big),\quad\text{for all } t\ge 0.$$

Repeating this arguments, we obtain $\hat{\Gamma}_N^{(n)}[0]-\gamma_N\preceq\underline{\Lambda}$ for all $n\ge 1$, and finally $\underline{L}^N-\gamma_N\preceq\underline{\Lambda}$. Hence, it holds with probability 1 that

$$\limsup_{N\to\infty}\underline{L}^N\preceq\underline{\Lambda}. \tag{15}$$

Take an arbitrary limit point ξ of a convergent subsequence, still denoted by $(\xi^N, N\ge 1)_{N\ge 1}$ for simplicity. It follows that for almost every realization ω,

$$\frac{1}{N}\sum_{i=1}^{N}\delta_{(\underline{X}^{N,i}(\omega),\underline{L}^N(\omega))}\to\xi(\omega)\text{ in }\mathcal{P}(\Xi).$$

On the other hand, there exists a solution (F,Λ) to Equation (3) such that $\xi(\omega)=\mathcal{L}(F,\Lambda)$. Therefore, $\underline{L}^N(\omega)\xrightarrow{\hat{d}}\Lambda$ and

$$\underline{\Lambda}_t\ge\lim_{N\to\infty}\underline{L}_t^N(\omega)=\Lambda_t\ge\underline{\Lambda}_t,\quad\text{for all } t\in J,$$

where $J\subset\mathbb{R}_+$ is the set of all points of continuity of L. Consequently, $\Lambda=\underline{\Lambda}$ and $\xi(\omega)=\mathcal{L}(\underline{X},\underline{\Lambda})$. In particular, the dominated convergence theorem yields the following:

$$\lim_{N\to\infty}\mathbb{E}[\underline{L}_t^N]=\underline{\Lambda}_t,\quad\text{for all } t\in J.$$

Set $c:=G(1)$. By symmetry, we get $\mathbb{P}(\tau_i^N=\infty)=\mathbb{P}(\tau_1^N=\infty)$ for all $1\le i\le N$. Thus,

$$\mathbb{E}\big[S_\infty^N\big]=\frac{1}{N}\sum_{i=1}^{N}\mathbb{P}(\tau_i^N=\infty)=\mathbb{P}(\tau_1^N=\infty).$$

Consider the two coupled processes below: for all $t\ge 0$,

$$X_t^{N,1}=Z^{N,1}+\beta t+B_t^1-G(L_t^N)\text{ and}$$
$$\hat{X}_t=(Z^{N,1}-\alpha(\gamma_N))+\beta t+B_t^1-G(\Lambda_t).$$

Set $\hat{\tau}:=\inf\{t\ge 0:\hat{X}_t\le 0\}$. It suffices to show $\lim_{N\to\infty}\mathbb{P}(\tau_1^N=\infty)=\mathbb{P}(\hat{\tau}=\infty)=\mathbb{P}(\tau=\infty)$. For $z\in\mathbb{R}$, let $p(z):=\mathbb{P}(z+\beta t+B_t^1>0,\ \forall t\ge 0)$. Then we have $\lim_{z\to\infty}p(z)=1$. For each $t\ge 0$, we get the following:

$$\begin{aligned}
&\Big|\mathbb{P}(\tau_1^N=\infty)-\mathbb{P}(\hat{\tau}=\infty)\Big|\\
&\le\Big|\mathbb{P}(\tau_1^N>t)-\mathbb{P}(\tau_1^N=\infty)\Big|\\
&\quad+\Big|\mathbb{P}(\tau_1^N>t)-\mathbb{P}(\hat{\tau}>t)\Big|\\
&\quad+\Big|\mathbb{P}(\hat{\tau}>t)-\mathbb{P}(\hat{\tau}=\infty)\Big|\\
&=\Big|\mathbb{P}(\tau_1^N>t)-\mathbb{P}(\tau_1^N=\infty)\Big|+\Big|\mathbb{E}[\underline{L}_t^N]-\underline{\Lambda}_t\Big|\\
&\quad+\Big|\mathbb{P}(\hat{\tau}>t)-\mathbb{P}(\hat{\tau}=\infty)\Big|,
\end{aligned}$$

where $\mathbb{P}(\tau_1^N>t)=1-\mathbb{E}[\underline{L}_t^N]$ is by symmetry. Furthermore,

$$\begin{aligned}
\Big|\mathbb{P}(\tau_1^N>t)-\mathbb{P}(\tau_1^N=\infty)\Big|&=\mathbb{E}[\mathbb{1}_{\{\tau_1^N>t\}}]-\mathbb{E}[\mathbb{1}_{\{\tau_1^N>t\}}\mathbb{1}_{\{\tau_1^N=\infty\}}]\\
&=\mathbb{E}[\mathbb{1}_{\{\tau_1^N>t\}}]\\
&\quad-\mathbb{E}\Big[\mathbb{P}(\tau_1^N=\infty|\tau_1^N>t)\mathbb{1}_{\{\tau_1^N>t\}}\Big].
\end{aligned}$$

On the event $\{\tau_1^N>t\}$, we have

$$\begin{aligned}
\{\tau_1^N=\infty\}&=\{X_t^{N,1}+\beta(s-t)+(B_s^1-B_t^1)-(G(L_s^N)-G(L_t^N))>0,\forall s\ge t\}\\
&\supset\{X_t^{N,1}-c+\beta(s-t)+(B_s^1-B_t^1)>0,\forall s\ge t\},
\end{aligned}$$

which yields by the Markov property $\mathbb{P}(\tau_1^N=\infty|\tau_1^N>t)\ge p(X_t^{N,1}-c)$, and thus

$$\begin{aligned}
\Big|\mathbb{P}(\tau_1^N>t)-\mathbb{P}(\tau_1^N=\infty)\Big|&=\mathbb{E}\Big[\mathbb{1}_{\{\tau_1^N>t\}}\big(1-\mathbb{P}(\tau_1^N=\infty|\tau_1^N>t)\big)\Big]\\
&\le\mathbb{E}\Big[\mathbb{1}_{\{\tau_1^N>t\}}\big(1-p(X_t^{N,1}-c)\big)\Big].
\end{aligned}$$

Set $Y_t:=Z^{N,1}+\beta t+B_t^1$ for all $t\ge 0$ and $\sigma:=\inf\{t\ge 0:Y_t\le 0\}$. Then $X_t^{N,1}\ge Y_t-c$ and $\tau_1^N\le\sigma$. Hence,

$$\begin{aligned}
\Big|\mathbb{P}(\tau_1^N>t)-\mathbb{P}(\tau_1^N=\infty)\Big|&\le\mathbb{E}\Big[\mathbb{1}_{\{\tau_1^N>t\}}\big(1-p(X_t^{N,1}-c)\big)\Big]\\
&\le\Big|\mathbb{E}\big[\mathbb{1}_{\{\sigma>t\}}(1-p(Y_t-2c))\big].
\end{aligned}$$

Similarly, we can establish $\Big|\mathbb{P}(\hat{\tau}>t)-\mathbb{P}(\hat{\tau}=\infty)\Big|\le\Big|\mathbb{E}[\mathbb{1}_{\{\sigma>t\}}(1-p(Y_t-\alpha(\gamma_N)-2c))]$, and thus

$$\begin{aligned}
\Big|\mathbb{P}(\tau_1=\infty)-\mathbb{P}(\hat{\tau}=\infty)\Big|&\le 2\mathbb{E}\big[\mathbb{1}_{\{\sigma>t\}}(1-p(Y_t-\alpha(1)-2c))\big]\\
&\quad+\Big|\mathbb{E}[\underline{L}_t^N]-\underline{\Lambda}_t\Big|.
\end{aligned}$$

As $\lim_{t\to\infty}Y_t=\infty$ holds almost surely, we have by the dominated convergence theorem,

$$\lim_{t\to\infty}\mathbb{E}\big[\mathbb{1}_{\{\sigma>t\}}(1-p(Y_t-\alpha(1)-2c))\big]=0.$$

For any $\varepsilon>0$, there exists t_ε such that $\mathbb{E}[\mathbb{1}_{\{\sigma>t\}}(1-p(Y(t)-\alpha(1)-2c))]\le\varepsilon$ for all $t\ge t_\varepsilon$. Fix an arbitrary $t\in J$ with $t>t_\varepsilon$. Then we get

$$\lim_{N\to\infty}\Big|\mathbb{P}(\tau_1^N=\infty)-\mathbb{P}(\hat{\tau}=\infty)\Big|\le 2\varepsilon+\lim_{N\to\infty}\Big|\mathbb{E}[\underline{L}_t^N]-\underline{\Lambda}_t\Big|=2\varepsilon,$$

which yields $\lim_{N\to\infty}\mathbb{E}[\underline{S}_\infty^N]=1-\underline{\Lambda}_\infty$. Take a sequence $(t_m)_{m\ge 1}\subset J$ satisfying $\lim_{m\to\infty}t_m=\infty$. Then,

$$\underline{S}_\infty^N=1-\underline{L}_\infty^N\le 1-\underline{L}_{t_m}^N\Longrightarrow\limsup_{N\to\infty}\underline{S}_\infty^N\le 1-\underline{\Lambda}_{t_m}.$$

Letting $m\to\infty$, we deduce that $\limsup_{N\to\infty}\underline{S}_\infty^N\le 1-\underline{\Lambda}_\infty$. On the other hand, we get by Fatou's lemma that $1-\underline{\Lambda}_\infty=\lim_{N\to\infty}\mathbb{E}[\underline{S}_\infty^N]\le\mathbb{E}[\limsup_{N\to\infty}\underline{S}_\infty^N]$, which implies $\limsup_{N\to\infty}\underline{S}_\infty^N=1-\underline{\Lambda}_\infty$. We conclude by applying Fatou's lemma,

$$\limsup_{N\to\infty}\mathbb{E}[|\underline{S}^N_\infty - 1 + \underline{\Lambda}_\infty|] \le \mathbb{E}\left[\limsup_{N\to\infty}|\underline{S}^N_\infty - 1 + \underline{\Lambda}_\infty|\right] = 0,$$

which yields the desired result. □

4.2 | Proof of Theorem 5

We aim to show that the regularized problem (5) is well-posed. Define the operators $\Gamma^\varepsilon : \mathcal{D} \to \mathcal{D}$ as follows: for $f \in D$, let $\Gamma^\varepsilon[f] \in \mathcal{D}$ be defined by the following:

$$\Gamma^\varepsilon[f]_t := 1 - \mathbb{E}\left[\exp\left(-\frac{1}{\varepsilon}\int_0^t (X^f_s)^- ds\right)\right],$$

where

$$X^f_t := Z + \beta t + W_t - G(f_t) \text{ and } \tau^f := \inf\{t \ge 0 : X^f_t \le 0\}.$$

It is easy to see that Γ^ε takes values in $C \cap \mathbb{M}$. By definition, $\ell^\varepsilon \in C \cap \mathbb{M}$ is a solution to Equation (5) if and only if ℓ^ε is a fixed point of Γ^ε. We start with a few technical results.

Lemma 13. *Let $f \in \mathcal{D}$ be nondecreasing and set*

$$A_t := \left\{\inf_{0\le s\le t}(W_s - f_s) = 0\right\}.$$

Then A_t is negligible for all $t > 0$.

Proof. Denote by $\tau := \inf\{s \ge 0 : W_s \le f_s\}$ the hitting time. Then it holds that

$$A_t = (A_t \cap \{\tau < t\}) \bigcup (A_t \cap \{\tau = t\}) =: B_t \cup C_t.$$

Clearly, $C_t \subset \{W_t = f_t\}$. As for B_t, we have the following:

$$\begin{aligned} B_t &\subset \{\tau < t \text{ and } W_\tau = f_\tau \text{ and } W_s \ge f_s \text{ for } s \in [\tau, t]\} \\ &\subset \{\tau < t \text{ and } W_\tau = f_\tau \text{ and } W_s \ge f_\tau \text{ for } s \in [\tau, t]\} := B'_t, \end{aligned}$$

where the second inclusion holds as f is nondecreasing. We conclude by the fact that there are infinite times $u \in [\tau, t]$ so that $W_u < W_\tau = f_\tau$. □

Lemma 14. *Let $(f^n)_{n\ge1} \subset \mathcal{D} \cap \mathbb{M}$ be a sequence such that $n \mapsto f^n_t$ is nondecreasing for all $t \ge 0$. Denote by f its pointwise limit. For every $x > 0$,*

$$\lim_{n\to\infty}\mathbb{P}\left(\inf_{0\le s\le t}(x + W_s - f^n_s) \le 0\right) = \mathbb{P}\left(\inf_{0\le s\le t}(x + W_s - f_s) \le 0\right),$$

holds for all $t \ge 0$.

Proof. Denote by τ_n (resp. τ) the first hitting time of $W - f^n$ (resp. $W - f$) at $-x$. Then,

$$\mathbb{P}\left(\inf_{0\le s\le t}(W_s - f^n_s) \le -x\right) = \mathbb{P}(\tau_n \le t) \text{ and } \mathbb{P}\left(\inf_{0\le s\le t}(W_s - f_s) \le -x\right) = \mathbb{P}(\tau \le t).$$

By assumption, we have $\tau \le \tau_n$, and thus

$$\begin{aligned} 0 \le \mathbb{P}(\tau \le t) - \mathbb{P}(\tau_n \le t) &= \mathbb{P}(\tau \le t, \tau_n > t) \\ &= \int_{(0,t]} \mathbb{P}(\tau_n > t | \tau = s)\mathbb{P}(\tau \in ds), \end{aligned}$$

where the last equality holds as $\tau > 0$. Restricted on the set $\{\tau = s\}$ for $s \in (0, t]$, we get

$$\{\tau_n > t\} \subset \{W_s - f^n_s > -x\} \cap \{W_s - f_s \le -x\}.$$

This implies the following:

$$\mathbb{P}(\tau_n > t|\tau = s) \le \mathbb{P}(f_s - x < W_s \le f^n_s - x) \to 0 \quad \text{as } n \to \infty,$$

and we conclude by the dominated convergence theorem. □

Now we prove Theorem 5.

Proof of Theorem 5. i. If the existence holds, then any solution is a strong solution. Let $X^\varepsilon, Y^\varepsilon$ be two arbitrary solutions of Equation (5). Observe that $|X^\varepsilon_t - Y^\varepsilon_t|$ is deterministic and satisfies the following:

$$|X^\varepsilon_t - Y^\varepsilon_t| \le C\mathbb{E}\left[\frac{1}{\varepsilon}\int_0^t |(X^\varepsilon_s)^- - (Y^\varepsilon_s)^-|ds\right] \le \frac{C}{\varepsilon}\int_0^t |X^\varepsilon_s - Y^\varepsilon_s|ds, \quad \text{for all } t \ge 0,$$

where $C > 0$ denotes the Lipschitz constant of G. This yields the uniqueness by Gronwall's inequality. To show the existence, we adopt the proof of the fixed-point theorem. Note that the operator Γ^ε preserves the monotonicity, that is, $f \preceq g \Longrightarrow \Gamma^\varepsilon[f] \preceq \Gamma^\varepsilon[g]$. Denote $\ell^0 \equiv 0$ and set for $n \ge 1$, $\ell^n := \Gamma^\varepsilon[\ell^{n-1}] \in C \cap \mathbb{M}$. Then we get by induction $\ell^{n-1} \preceq \ell^n$ for all $n \ge 1$. Hence, $n \mapsto \ell^n_t$ is nonincreasing for all $t \ge 0$, and the pointwise $\ell_t := \lim_{n\to\infty} \ell^n_t \in [0, 1]$ exists. Finally, rewriting the equality

$$\ell^{n+1}_t = G\left(1 - \mathbb{E}\left[\exp\left(-\frac{1}{\varepsilon}\int_0^t (Z + \beta s + W_s - \alpha\ell^n_s)^- ds\right)\right]\right), \quad \text{for all } t \ge 0,$$

we conclude the desired existence by the dominated convergence theorem.

ii. Fix arbitrary $\delta < \varepsilon$. We claim that $\ell^\delta \preceq \ell^\varepsilon$. Indeed, it is known from the proof above that

$$\ell^\delta = \lim_{n\to\infty}\Gamma^{\delta,n}[\ell^0] \text{ and } \ell^\varepsilon = \lim_{n\to\infty}\Gamma^{\varepsilon,n}n[\ell^0],$$

where $\Gamma^{\delta,n}$ (resp. $\Gamma^{\varepsilon,n}$) is the n-composition of Γ^δ (resp. Γ^ε). By definition, we have $\Gamma^\varepsilon(f) \preceq \Gamma^\delta(f)$ for all $f \in \mathcal{D}$. In particular, $\Gamma^\varepsilon(\ell^0) \preceq \Gamma^\delta(\ell^0)$. Similar to the argument in the proof of Theorem 5 (ii), we deduce the following:

$$\Gamma^{\varepsilon,n+1}[\ell^0] = \Gamma^\varepsilon\left[\Gamma^{\varepsilon,n}[\ell^0]\right] \preceq \Gamma^\varepsilon\left[\Gamma^{\delta,n}[\ell^0]\right] \preceq \Gamma^\delta\left[\Gamma^{\delta,n}[\ell^0]\right] = \Gamma^{\delta,n+1}[\ell^0],$$

where the second inequality follows from the induction and the monotonicity of Γ^ε. Therefore, we get $\Gamma^{\varepsilon,n}[\ell^0] \preceq \Gamma^{\delta,n}[\ell^0]$ for every $n \geq 1$, and $\ell^\varepsilon \leq \ell^\delta$. Namely, for each $t \geq 0$, $[0,1] \ni \varepsilon \mapsto \ell^\varepsilon_t \in [0,1]$ is nonincreasing, and we can define the pointwise limit $\hat{\ell}_t := \lim_{\varepsilon \to 0+} \ell^\varepsilon_t$. Thanks to Proposition 15, $\hat{\ell}$ is a solution to Equation (3).

Proposition 15. *$\hat{\ell}$ is a solution to the McKean–Vlasov equation (3).*

Proof. Set $\hat{X}_t := Z + \beta t + W_t - \alpha \hat{\ell}_t$ for $t \geq 0$. By definition, we have

$$X^\varepsilon_t = Z + \beta t + W_t - G\left(1 - \mathbb{E}\left[\exp\left(-\frac{1}{\varepsilon}\int_0^t (X^\varepsilon_s)^- ds\right)\right]\right)$$

and

$$\begin{aligned}
&\left|X^\varepsilon_t - \left(Z + \beta t + W_t - G\left(\mathbb{P}(\inf_{0\leq s\leq t} X^\varepsilon_s \leq 0)\right)\right)\right| \\
&= \left|G\left(1 - \mathbb{E}\left[\exp\left(-\frac{1}{\varepsilon}\int_0^t (X^\varepsilon_s)^- ds\right)\right]\right) - G\left(\mathbb{E}[\mathbb{1}_{\{\inf_{0\leq s\leq t} X^\varepsilon_s \leq 0\}}]\right)\right| \\
&\leq C\left|1 - \mathbb{E}\left[\exp\left(-\frac{1}{\varepsilon}\int_0^t (X^\varepsilon_s)^- ds\right)\right] - \mathbb{E}[\mathbb{1}_{\{\inf_{0\leq s\leq t} X^\varepsilon_s \leq 0\}}]\right| \\
&= C\left|\mathbb{E}\left[\mathbb{1}_{\{\inf_{0\leq s\leq t} X^\varepsilon_s > 0\}} - \exp\left(-\frac{1}{\varepsilon}\int_0^t (X^\varepsilon_s)^- ds\right)\right]\right| \\
&= C\mathbb{E}\left[\exp\left(-\frac{1}{\varepsilon}\int_0^t (X^\varepsilon_s)^- ds\right)\mathbb{1}_{\{\inf_{0\leq s\leq t} X^\varepsilon_s < 0\}}\right],
\end{aligned}$$

where the last equality follows from Lemma 13. Applying the dominated convergence theorem, we get the following:

$$\lim_{\varepsilon \to 0+} \mathbb{E}\left[\exp\left(-\frac{1}{\varepsilon}\int_0^t (X^\varepsilon_s)^- ds\right)\mathbb{1}_{\{\inf_{0\leq s\leq t} X^\varepsilon_s < 0\}}\right] = 0$$

and thus $\hat{X}_t = Z + \beta t + W_t - \alpha \lim_{\varepsilon \to 0+} \mathbb{P}[\inf_{0\leq s\leq t} X^\varepsilon_s \leq 0]$, which yields the desired result by Lemma 14. □

Recall that $\underline{\Lambda}$ is the minimal solution to Equation (3). Then it suffices to show $\ell^\varepsilon \leq \underline{\Lambda}$ for every $\varepsilon > 0$. Recall that $\underline{\Lambda} = \lim_{n\to\infty} \Gamma_n[\ell^0]$. Again, we prove $\Gamma^\varepsilon[f] \leq \Gamma[f]$ for all $f \in \mathcal{D}$. Indeed, we rewrite as follows:

$$\Gamma^\varepsilon[f]_t = G\left(\mathbb{E}\left[1 - \exp\left(-\frac{1}{\varepsilon}\int_0^t (X^f_s)^- ds\right)\right]\right) \text{ and}$$

$$\Gamma[f]_t = G\left(\mathbb{E}[\mathbb{1}_{\{\inf_{0\leq s\leq t} X^f_s \leq 0\}}]\right)$$

and then compare the two random variables by distinguishing the following cases:

- On $\{\inf_{0\leq s\leq t} X^f_s \leq 0\}$, it holds the following:

$$1 - \exp\left(-\frac{1}{\varepsilon}\int_0^t (X^f_s)^- ds\right) \leq 1 = \mathbb{1}_{\{\inf_{0\leq s\leq t} X^f_s \leq 0\}};$$

- On $\{\inf_{0\leq s\leq t} X^f_s > 0\}$, it holds the following:

$$1 - \exp\left(-\frac{1}{\varepsilon}\int_0^t (X^f_s)^- ds\right) = 0 = \mathbb{1}_{\{\inf_{0\leq s\leq t} X^f_s \leq 0\}}.$$

This yields in particular $\Gamma^\varepsilon[\ell^0] \leq \Gamma[\ell^0]$. Then we have the following:

$$\Gamma^{\varepsilon,n+1}[\ell^0] = \Gamma^\varepsilon\left[\Gamma^{\varepsilon,n}[\ell^0]\right] \leq \Gamma^\varepsilon\left[\Gamma^n[\ell^0]\right] \leq \Gamma\left[\Gamma^n[\ell^0]\right] = \Gamma^{n+1}[\ell^0],$$

where the second inequality follows from the induction and the monotonicity of Γ^ε. Letting $n \to \infty$ and then $\delta \to 0$, we obtain $\hat{\ell} \leq \underline{\Lambda}$, which implies that $\hat{\ell} = \underline{\Lambda}$ is the minimal solution of Equation (3). □

5 | Proof of Theorem 6

In this section, we analyze the effect of capital provision. Following the same lines as in the proof of Theorem 2, we can show that Equation (6) has a unique minimal solution, which yields its well-posedness. In what follows, we distinguish the cases $\beta < 0$, $\beta = 0$, and $\beta > 0$.

5.1 | Case of $\beta < 0$

Proof of Theorem 6. (i) For each $\phi \in \Phi_N$, let $(X^{\phi,N,1}, \dots, X^{\phi,N,N}, L^{\phi,N})$ be an arbitrary solution to Equation (6). For each $t > 0$, we have for $i = 1, \dots, N$,

$$\{\tau^\phi_i > t\} = \left\{\tau^\phi_i > t, \int_0^t \phi^i_s ds > -\frac{\beta t}{2}\right\} \bigcup \left\{\tau^\phi_i > t, \int_0^t \phi^i_s ds \leq -\frac{\beta t}{2}\right\}.$$

Hence,

$$\begin{aligned}
\sum_{i=1}^N \mathbb{1}_{\{\tau^\phi_i > t\}} &\leq \sum_{i=1}^N \mathbb{1}_{\{\tau^\phi_i > t, \int_0^t \phi^i_s ds > -\frac{\beta t}{2}\}} + \sum_{i=1}^N \mathbb{1}_{\{\tau^\phi_i > t, \int_0^t \phi^i_s ds \leq -\frac{\beta t}{2}\}} \\
&\leq -\frac{2}{\beta} + \sum_{i=1}^N \mathbb{1}_{\{\tau^\phi_i > t, \int_0^t \phi^i_s ds \leq -\frac{\beta t}{2}\}}.
\end{aligned}$$

On the event $\{\tau^\phi_i > t, \int_0^t \phi^i_s ds \leq -\beta t/2\}$, we get the following:

$$X^{\phi,i}_t \leq Z^i + \frac{\beta t}{2} + B^i_t, \quad \text{for all } t \geq 0,$$

and thus

$$\left\{\tau^\phi_i > t, \int_0^t \phi^i_s ds \leq -\frac{\beta t}{2}\right\} \subset \left\{Z^i + B^i_t > -\frac{\beta t}{2}\right\}.$$

Consequently,

$$\mathbb{E}[S^{\phi,N}_\infty] = \lim_{t\to\infty} \mathbb{E}\left[\sum_{i=1}^N \mathbb{1}_{\{\tau^\phi_i > t\}}\right] \leq -\frac{2}{\beta} + \lim_{t\to\infty} N\mathbb{P}(Z + B_t > -\beta t/2) = -\frac{2}{\beta}.$$

□

5.2 | Case of $\beta = 0$

Theorem 6 (ii) asserts that at criticality where $\beta = 0$, the number of surviving banks scales as $\sqrt{N}$ if the initial capital levels $X_0^{N,i}$ are i.i.d. according to θ^N, provided that θ is compactly supported. For simplicity, we assume that $\theta^N = \delta_1$ without loss of generality. Since the model (6) is stochastically bounded from above by the up-the-river model via a natural coupling, the upper bound in Equation (31) follows easily from Tang and Tsai (2018). The main task is to establish the lower bound, which boils down to a few lemmas.

Let $\tau_{(k)}^N$ be the first time that k entities go bankrupt, that is, k members of $X^{N,1}, \ldots X^{N,N}$ entering $(-\infty, 0]$. The following lemma implies that a significant fraction of banks being ruined cannot occur in too short time.

Lemma 16. *Let $\varepsilon > 0$ such that $\alpha\varepsilon < 1$, and*

$$T_{\alpha,\varepsilon} := \left(\frac{1-\alpha\varepsilon}{\widetilde{\mathcal{N}}^{-1}\left(\frac{1}{2}\varepsilon(1-\varepsilon)^{\frac{1-\varepsilon}{\varepsilon}}\right)} \right)^2, \tag{16}$$

where $\widetilde{\mathcal{N}}(x) := \mathbb{P}(B_1 > x)$ is the tail distribution of standard normal. Then for $T < T_{\alpha,\varepsilon}$, there is $C > 0$ such that

$$\mathbb{P}\left(\tau_{(\varepsilon N)}^N < T\right) \le CN^{-k}, \tag{17}$$

for any $k > 0$, as $N \to \infty$.

Proof. Let $\widetilde{X}_t^{N,i} := (1-\alpha\varepsilon) + B_t^i$ be driven by the same Brownian motion as $X^{N,i}$, and $\widetilde{\tau}_{(k)}^N$ be the k^{th} hitting time of $\{\widetilde{X}^{N,1}, \ldots, \widetilde{X}^{N,N}\}$ to 0. By the coupling of $\widetilde{X}^N$ and X^N, we get for $0 \le t \le \tau_{(\varepsilon N)}^N$,

$$\frac{1}{N}\sum_{k=1}^{N} 1_{\{\tau_k^N \le t\}} \le \varepsilon \quad \text{and} \quad X_t^{N,i} = 1 + B_t^i - \frac{\alpha}{N}\sum_{k=1}^{N} 1_{\{\tau_k^N \le t\}} \ge \widetilde{X}_t^{N,i}.$$

Consequently,

$$\mathbb{P}\left(\tau_{(\varepsilon N)}^N < T\right) \le \mathbb{P}\left(\widetilde{\tau}_{(\varepsilon N)}^N < T\right). \tag{18}$$

We proceed to bounding $\mathbb{P}\left(\widetilde{\tau}_{(\varepsilon N)}^N < T\right)$. Let $\mathcal{S}_\varepsilon$ be the set of (εN)-tuples of $\{1, \ldots, N\}$, so $\#\mathcal{S}_\varepsilon = \binom{N}{\varepsilon N}$ We have the following:

$$\begin{aligned} \mathbb{P}\left(\widetilde{\tau}_{(\varepsilon N)}^N < T\right) &= \mathbb{P}\left(\bigcup_{S \in \mathcal{S}_\varepsilon} \left\{ \max_{k \in S} \widetilde{\tau}_k^N < T \right\} \right) \\ &\le \binom{N}{\varepsilon N} \mathbb{P}\left(\max_{k \le \varepsilon N} \widetilde{\tau}_k^N < T \right) = \binom{N}{\varepsilon N} \mathbb{P}\left(\widetilde{\tau}_1^N < T\right)^{\varepsilon N}, \end{aligned} \tag{19}$$

where the second inequality is by the union bound, and the third equality follows from the fact that $\widetilde{X}^{N,i}$ are i.i.d. Note that

$$\begin{aligned} \mathbb{P}(\widetilde{\tau}_1^N < T) &= \mathbb{P}\left(\inf_{0 \le t \le T} B_t^i \le \alpha\varepsilon - 1 \right) = \mathbb{P}(|B_T| > 1 - \alpha\varepsilon) \\ &= 2\widetilde{\mathcal{N}}\left(\frac{1-\alpha\varepsilon}{\sqrt{T}} \right), \end{aligned}$$

where the second equality is by the reflection principle $\inf_{0 \le t \le T} B_t^i \overset{d}{=} -|B_T^i|$ (see Karatzas and Shreve 1991, Section 2.8A). As a result,

$$\mathbb{P}\left(\widetilde{\tau}_1^N < T\right)^{\varepsilon N} = \exp\left(N\varepsilon \log\left(2\widetilde{\mathcal{N}}\left(\frac{1-\alpha\varepsilon}{\sqrt{T}} \right) \right) \right). \tag{20}$$

Moreover,

$$\binom{N}{\varepsilon N} \sim \exp\left(N(-\varepsilon\log\varepsilon - (1-\varepsilon)\log(1-\varepsilon))\right), \quad \text{as } N \to \infty, \tag{21}$$

where we only focus on the exponential, and neglect the polynomial factor of N in Equation (21). Combining Equations (19)–(21) yields

$$\mathbb{P}\left(\widetilde{\tau}_{(\varepsilon N)}^N < T\right) \lesssim \exp\left(N h_{\alpha,\varepsilon}(T)\right), \tag{22}$$

where

$$h_{\alpha,\varepsilon}(T) := \varepsilon \log\left(2\widetilde{\mathcal{N}}\left(\frac{1-\alpha\varepsilon}{\sqrt{T}} \right) \right) - \varepsilon\log\varepsilon - (1-\varepsilon)\log(1-\varepsilon). \tag{23}$$

Observe that $T \to h_{\alpha,\varepsilon}(T)$ is increasing on $(0, \infty)$, and

$$\lim_{T \to 0} h_{\alpha,\varepsilon}(T) = -\infty, \quad \lim_{T \to \infty} h_{\alpha,\varepsilon}(T) = -\varepsilon\log\varepsilon - (1-\varepsilon)\log(1-\varepsilon) > 0,$$

so the unique root of $h_{\alpha,\varepsilon}(T) = 0$ on $(0, \infty)$ is $T = T_{\alpha,\varepsilon}$ defined by Equation (16). Therefore, $h_{\alpha,\varepsilon}(T) < 0$ for $T < T_{\alpha,\varepsilon}$, and the bound (17) follows from Equations (18) and (22). □

The next lemma shows that many (most) of the banks surviving up to time $T_{\alpha,\varepsilon}$ have suitably large capital levels.

Lemma 17. *Let $\delta > 0$, and let*

$$Z_{\alpha,\varepsilon,\delta} := \#\left\{ k : X_{T_{\alpha,\varepsilon}}^{N,k} \ge \alpha + \delta \text{ and } X_t^{N,k} > 0 \text{ for } t \le T_{\alpha,\varepsilon} \right\}, \tag{24}$$

be the number of surviving banks whose final capital levels are above $\alpha + \delta$ at time $T_{\alpha,\varepsilon}$. Let

$$p_{\alpha,\varepsilon,\delta} := \widetilde{\mathcal{N}}\left(\left(\frac{\alpha+\delta}{1-\alpha\varepsilon} - 1 \right) \widetilde{\mathcal{N}}^{-1}\left(\frac{1}{2}\varepsilon(1-\varepsilon)^{\frac{1-\varepsilon}{\varepsilon}} \right) \right). \tag{25}$$

Then we have the following:

$$\mathbb{E}Z_{\alpha,\varepsilon,\delta} \ge (1-\varepsilon)p_{\alpha,\varepsilon,\delta}N, \tag{26}$$

and for $\lambda < (1-\varepsilon)p_{\alpha,\varepsilon,\delta}$,

$$\mathbb{P}(Z_{\alpha,\varepsilon,\delta} > \lambda N) \ge 1 - CN^{-k}, \tag{27}$$

for any $k > 0$, as $N \to \infty$.

Proof. Let $\Omega_{\alpha,\varepsilon} := \left\{\tau^N_{(\varepsilon n)} > T_{\alpha,\varepsilon}\right\}$. By Lemma 16, we get $\mathbb{P}(\Omega^c_{\alpha,\varepsilon}) \lesssim N^{-2}$. Then we have the following:

$$\begin{aligned}
\mathbb{E}Z_{\alpha,\varepsilon,\delta} &= \mathbb{E}\left(Z_{\alpha,\varepsilon,\delta}1_{\Omega_{\alpha,\varepsilon}}\right) + \mathbb{E}\left(Z_{\alpha,\varepsilon,\delta}1_{\Omega^c_{\alpha,\varepsilon}}\right) \\
&= \mathbb{E}\left(Z_{\alpha,\varepsilon,\delta} \mid \Omega_{\alpha,\varepsilon}\right)(1-\mathbb{P}(\Omega^c_{\alpha,\varepsilon})) + \mathbb{E}\left(Z_{\alpha,\varepsilon,\delta}1_{\Omega^c_{\alpha,\varepsilon}}\right) \\
&= \mathbb{E}\left(Z_{\alpha,\varepsilon,\delta} \mid \Omega_{\alpha,\varepsilon}\right) + \mathcal{O}(N \cdot N^{-2}) \\
&= \mathbb{E}\left(Z_{\alpha,\varepsilon,\delta} \mid \Omega_{\alpha,\varepsilon}\right) + \mathcal{O}(N^{-1}),
\end{aligned} \tag{28}$$

where we use the fact that $Z_{\alpha,\varepsilon,\delta} \le N$ and $\mathbb{P}(\Omega^c_{\alpha,\varepsilon}) \lesssim N^{-2}$ in the third equation. On the event $\Omega_{\alpha,\varepsilon}$, there are at least $(1-\varepsilon)N$ surviving banks, the capital levels of which are stochastically larger than Brownian motion starting at $1-\alpha\varepsilon$ (see Day 1983). By an obvious coupling (to Brownian particles without control ϕ), $(Z_{\alpha,\varepsilon,\delta} \mid \Omega_{\alpha,\varepsilon})$ is stochastically larger than $\mathrm{Bin}((1-\varepsilon)N, p_{\alpha,\varepsilon,\delta})$, where

$$p_{\alpha,\varepsilon,\delta} := \mathbb{P}\left(B_{T_{\alpha,\varepsilon}} \ge \alpha(1+\varepsilon)+\delta-1\right) = \widetilde{\mathcal{N}}\left(\frac{\alpha(1+\varepsilon)+\delta-1}{\sqrt{T_{\alpha,\varepsilon}}}\right). \tag{29}$$

Recall the definition of $T_{\alpha,\varepsilon}$ from Equation (16), and the expression (29) simplifies to the right side of Equation (25). Combining with Equation (28) yields the lower bound (26). The bound (27) follows from the stochastic comparison and the Chernorff bound. ◻

Now we prove Theorem 6 (ii) .

Proof of Theorem 6. (ii) The difference between Equation (6) and the up-the-river model is that there is an additional term $-\frac{\alpha}{N}\sum_{k=1}^N 1_{\{\tau_k^N \le t\}}$, which drags the capital level down. Under the natural coupling of the model (6) and the up-the-river model, if a bank survives using a control in the model (6), it will also survive using the same control in the up-the-river model. This implies the following:

$$\limsup_{N\to\infty} \frac{S^{\phi,N}_{\infty}}{\sqrt{N}} \le \limsup_{N\to\infty} \frac{S^{\mathrm{UR}}_N}{\sqrt{N}},$$

where S^{UR}_N denotes the maximum number of surviving banks in the up-the-river model. By Tang and Tsai (2018, Theorem 1.2), we have $\lim_{N\to\infty} \frac{S^{\mathrm{UR}}_N}{\sqrt{N}} = 4/\sqrt{\pi}$, and hence the upper bound in Equation (31). Now we prove the lower bound. Fix $m > 0$. We use the following control strategy:

- For each bank, exercise no control until it reaches the capital level N/m.
- If the bank does reach the capital level N/m, and is the one of the first m banks achieving so, then exercise the control $1/m$ thereafter.

We write $S^{\phi,N} \equiv S_N$ without any danger of confusion. For $\lambda < (1-\varepsilon)p_{\alpha,\varepsilon,\delta}$, let $\Omega_{\alpha,\varepsilon,\delta,\lambda} := \{Z_{\alpha,\varepsilon,\delta} > \lambda N\}$. By Lemma 17, we have $\mathbb{P}(\Omega_{\alpha,\varepsilon,\delta,\lambda}) \ge 1 - N^{-2}$. On the event $\Omega_{\alpha,\varepsilon,\delta,\lambda}$, there are (at least) λN banks whose capital levels are above $\alpha+\delta$ at time $T_{\alpha,\varepsilon}$. The capital levels of these banks after $T_{\alpha,\varepsilon}$ are stochastically larger than Brownian motion starting at δ. Recall from Durrett (2019, Theorem 7.5.3) the following:

$$\mathbb{P}\left(\delta + B_t \text{ hits } \frac{N}{m} \text{ before } 0\right) = \frac{\delta m}{N}.$$

As a result, the number of banks whose capital levels reach K/m is bounded from below by the following:

$$\min\left(m, \mathrm{Bin}(\lambda N, \frac{\delta m}{N})\right).$$

Also note the following (Durrett 2019, Exercise 7.5.2):

$$\mathbb{P}\left(\frac{N}{m} - \alpha + B_t + \frac{t}{m} \text{ does not hit } 0\right) = 1 - \exp\left(-\frac{2}{m}\left(\frac{N}{m}-\alpha\right)\right).$$

Therefore,

$$\begin{aligned}
\mathbb{E}S_N &\ge \left[1-\exp\left(-\frac{2}{m}\left(\frac{N}{m}-\alpha\right)\right)\right] \\
&\quad \mathbb{E}\left[\min\left(m, \mathrm{Bin}\left(\lambda N, \frac{\delta m}{N}\right)\right)\right] + \mathcal{O}(N^{-1}) \\
&\overset{m=\theta\sqrt{N}}{=} \min(1,\lambda\delta)\,\theta\left(1-\exp\left(-\frac{2}{\theta^2}\right)\right)\sqrt{N} + o(1).
\end{aligned}$$

By taking $\lambda \uparrow (1-\varepsilon)p_{\alpha,\varepsilon,\delta}$, we get the following:

$$\liminf_{N\to\infty} \frac{\mathbb{E}S_N}{\sqrt{N}} \ge \min\left(1, (1-\varepsilon)\delta p_{\alpha,\varepsilon,\delta}\right) \sup_{\theta}\left\{\theta\left(1-\exp\left(-\frac{2}{\theta^2}\right)\right)\right\}. \tag{30}$$

Note that $\sup_\theta\left\{\theta\left(1-\exp\left(-\frac{2}{\theta^2}\right)\right)\right\} \approx 0.9$, which is attained at $\theta \approx 1.26$. Further by taking $\varepsilon = \min\left(\frac{1}{2}, \frac{1}{2\alpha}\right)$ and $\delta = \alpha$ yields the following:

$$c_\alpha \le \liminf_{N\to\infty} \frac{\mathbb{E}S_N}{\sqrt{N}} \le \limsup_{N\to\infty} \frac{\mathbb{E}S_N}{\sqrt{N}} \le \frac{4}{\sqrt{\pi}}, \tag{31}$$

where $c_\alpha \approx 0.9\min(1, \rho_\alpha)$ with

$$\rho_\alpha := \begin{cases} \left(\alpha - \frac{1}{2}\right)\widetilde{\mathcal{N}}\left((4\alpha-1)\widetilde{\mathcal{N}}^{-1}\left(\frac{1}{4\alpha}\left(1-\frac{1}{2\alpha}\right)^{2\alpha-1}\right)\right) & \text{for} \quad \alpha > 1 \\ \frac{1}{2}\alpha\widetilde{\mathcal{N}}\left(\frac{5\alpha-2}{2-\alpha}\,\widetilde{\mathcal{N}}^{-1}(1/8)\right) & \text{for} \quad \alpha \le 1. \end{cases} \tag{32}$$

◻

Note that the lower and upper bounds appearing in the proof above may not be optimal. It is easy to see from the proof that the lower bound can be numerically improved to $c'_\alpha \approx 0.9\min(1, \rho'_\alpha)$ with

$$\rho'_\alpha := \sup_{\varepsilon,\delta}\left\{(1-\varepsilon)\delta p_{\alpha,\varepsilon,\delta}\right\}.$$

But it does not seem that ρ'_α is easily simplified, and has a closed-form expression. It is also possible to refine the stochastic comparisons in Lemmas 16 and 17 to further improve the bounds in Equation (31).

5.3 | Case of $\beta > 0$

Finally, we consider the positive economy where the capital level of each bank has a positive drift $\beta > 0$:

$$X_t^{N,i} = X_0^{N,i} + \beta t + B_t^i - \frac{\alpha}{N}\sum_{k=1}^N 1_{\{\tau_k^N \le t\}}. \tag{33}$$

The question is whether the capital provision is helpful, and what is the optimal allocation to maximize the number of surviving banks. As previously mentioned, this problem is challenging.

To provide insights, we focus on the setting without interaction, that is, $\alpha = 0$. In this case, the equations

$$X_t^{\phi,N,i} = X_0^{N,i} + \beta t + B_t^i + \int_0^t \phi_s^{N,i} ds, \tag{34}$$

have a unique solution, and we look for

$$S_N = \max_\phi S^{\phi,N} := \max_\phi \sum_{k=1}^N 1_{\{\tau_k^{\phi,N} = \infty\}}$$

where $\tau_k^{\phi,N} := \inf\{t > 0 : X_t^{\phi,N,i} \le 0\}$. For $\phi = 0$, we have $X_t^{0,N,i} = X_0^{N,i} + \beta t + B_t^i$, which are i.i.d. Brownian motion with drift $\beta > 0$. By Durrett (2019, Exercise 7.5.2),

$$\mathbb{P}(X_0^{N,i} + \beta t + B_t^i \text{ does not hit } 0) = 1 - \mathbb{E}\left[e^{-2\beta X_0^{N,i}}\right]. \tag{35}$$

Thus, $\mathbb{E}S^{0,N} = \left(1 - \mathbb{E}\left[e^{-2\beta X_0^{N,i}}\right]\right)N$. We aim to prove that in the supercritical case where $\beta > 0$, the number of surviving banks scales as N, and moreover, the unit control does not help. That is, for any $\phi \in \Phi_N$,

$$\lim_{N\to\infty} \frac{\mathbb{E}S^{\phi,N}}{N} = \lim_{N\to\infty} \frac{\mathbb{E}S^{0,N}}{N} = 1 - \mathbb{E}\left[e^{-2\beta X_0^{N,1}}\right].$$

To this end, we need the following lemma.

Lemma 18. *Let $Y_t^x = x + \beta t + B_t$, and $\Psi(t,x) := \mathbb{P}(\inf_{s\le t} Y_s^x > 0)$. Then*

1. *$\Psi(t,x)$ solves the PDE*

$$\frac{\partial \Psi}{\partial t} = \beta\frac{\partial \Psi}{\partial x} + \frac{1}{2}\frac{\partial^2 \Psi}{\partial t^2}, \quad \Psi(t,0) = 0, \quad \Psi(0,x) = 1_{\{x>0\}}. \tag{36}$$

2. *We have*

$$\Psi(t,x) = \Phi\left(\frac{x+\beta t}{\sqrt{t}}\right) - e^{-2\beta x}\Phi\left(\frac{-x+\beta t}{\sqrt{t}}\right), \tag{37}$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-\frac{z^2}{2}}dz$ is the cumulative distribution function of standard normal.

Proof.

1. Let $T_x := \inf\{t > 0 : Y_t^x = 0\}$ be the first time at which Y^x hits 0. By duality, we have the following:

$$\Psi(t,x) := \mathbb{P}\left(\inf_{s\le t} Y_s^x > 0\right) = \mathbb{P}(T_x > t),$$

which is the Green function of Y^x absorbed at 0. It follows from the classical Feynman–Kac formula (see Gardiner 2004, Section 5.4) that Ψ solves the initial-boundary problem (36).

2. It is easy to check that the formula (37) satisfies the PDE (36). It can also be read from Buckholtz and Wasan (1976) and Cox and Miller (1965, Example 5.5).

□

Proof of Theorem 6. (iii) First by the natural coupling, we have for any control $\phi \in \Phi_N$,

$$\mathbb{E}S^{\phi,N} \ge \mathbb{E}S^{0,N} = \left(1 - \mathbb{E}\left[e^{-2\beta X_0^{N,1}}\right]\right)N.$$

This implies that $\liminf_{N\to\infty} \mathbb{E}S_N/N \ge 1 - \mathbb{E}\left[e^{-2\beta X_0^{N,1}}\right]$. Recall the definition of $\Psi(t,x)$ in Lemma 18. Let $Z_s = \sum_{i=1}^N \Psi(t - s, X_s^{\phi,N,i})$, $s \le t$. Applying Itô's formula, we get the following:

$$\begin{aligned} Z_t &= N\Psi(t, X_0^{N,i}) + \sum_{i=1}^N \int_0^t \left(-\frac{\partial}{\partial t} + \beta\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\right)\Psi(t-s, X_s^{\phi,N,i})ds \\ &\quad + \int_0^t \sum_{i=1}^N \phi^{N,i}(s)\frac{\partial}{\partial x}\Psi(t-s, X_s^{\phi,N,i})ds + \text{martingale} \\ &= N\Psi(t, X_0^{N,i}) + \int_0^t \sum_{i=1}^N \phi^{N,i}(s)\frac{\partial}{\partial x}\Psi(t-s, X_s^{\phi,N,i})ds + \text{martingale}, \end{aligned} \tag{38}$$

where the second equality follows from Equation (36). Next by Equation (37), we get the following:

$$\frac{\partial \Psi}{\partial x}(t,x) = p(t, x+\beta t) + e^{-2\beta x}p(t, x-\beta t) + 2\beta e^{-2\beta x}\Phi\left(\frac{-x+\beta t}{\sqrt{t}}\right), \tag{39}$$

where $p(t,x) := \frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{x^2}{2t}\right)$ is the heat kernel. Observe that for $x \ge 0$,

$$p(t, x+\beta t) \lesssim t^{-\frac{1}{2}}e^{-\beta^2 t}, \qquad e^{-2\beta x}p(t, x-\beta t) \lesssim t^{-\frac{1}{2}}e^{-\beta^2 t},$$
$$e^{-2\beta x}\Phi\left(\frac{-x+\beta t}{\sqrt{t}}\right) \le 1. \tag{40}$$

By Equations (39) and (40), we get the following:

$$\int_0^t \sum_{i=1}^N \phi^{N,i}(s)\frac{\partial}{\partial x}\Psi(t-s, X_s^{\phi,N,i})ds \le C\int_0^t \frac{e^{-\beta^2(t-s)}}{\sqrt{t-s}}ds + 2\beta t \le C' + 2\beta t. \tag{41}$$

Combining Equations (38) and (41) yields $\mathbb{E}S_N \le \mathbb{E}Z_t \le N\mathbb{E}\Psi(t, X_0^{N,1}) + 2\beta t + C'$ for any $t > 0$. Fixing $\varepsilon > 0$ and taking

$t = \frac{\varepsilon}{2\beta}N$, we get the following:

$$\limsup_{N\to\infty} \frac{\mathbb{E}S_N}{N} \le \lim_{N\to\infty} \mathbb{E}\Psi\left(\frac{\varepsilon N}{2\beta}, X_0^{N,1}\right) + \varepsilon = 1 - \mathbb{E}\left[e^{-2\beta X_0^{N,1}}\right] + \varepsilon. \quad (42)$$

Since Equation (42) holds for any $\varepsilon > 0$, we conclude that $\limsup_{N\to\infty} \mathbb{E}S_N/N \le 1 - \mathbb{E}\left[e^{-2\beta X_0^{N,1}}\right]$. □

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Data Availability Statement

The authors have nothing to report.

Endnotes

[1] In this paper, we do not distinguish between the "banks" and the "particles," if there is no confusion. Note that the word "particle" may have various interpretations under different models, for example, "component," "agent," "cell," and so forth.

[2] The existing research focuses on the case $\beta = 0$ and $G(x) = \alpha x$ for $\alpha > 0$, while Nadtochiy and Shkolnikov (2019) consider the alternative case $G(x) = -\alpha \log(1 - x)$ for $\alpha > 0$.

[3] In these papers, the central planner refers to the government. It is worth noting that equity injection in banks by the government is a very exceptional situation.

[4] In practice, the capital provision is added in discrete time, that is, the planner is provided a budget of $1 \times dt$ during the periods $[0, dt], [dt, 2dt], [2dt, 3dt]$, and so forth. Here we take $dt \to 0$, and the constraint is on the capital provision rate. In a finite time horizon T, the unit capital provision rate yields a total budget of T.

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