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THE RECALIBRATION CONUNDRUM: HEDGING VALUATION ADJUSTMENT FOR CALLABLE CLAIMS

CYRIL BÉNÉZET

Laboratoire de Mathématiques et Modélisation d'Evry (LaMME)
Université Paris-Saclay, Univ. Evry, ensIIE, CNRS
91037 Evry-Courcouronnes, France
cyril.benezet@ensiie.fr

STÉPHANE CRÉPEY* and DOUNIA ESSAKET†

Laboratoire de Probabilités
Statistique et Modélisation (LPSM)
Université Paris Cité
CNRS UMR 8001, Paris, France
**stephane.crepey@lpsm.paris; screpey3@gmail.com*
†essaket@lpsm.paris

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The dynamic hedging theory only makes sense in the setup of one given model, whereas the practice of dynamic hedging is just the opposite, with models fleeing after the data through daily recalibration. This is quite a quantitative finance paradox. In this paper, we revisit the notion of hedging valuation adjustment (HVA), originally intended to deal with dynamic hedging frictions, in the direction of recalibration and model risks. Specifically, we extend to callable assets the HVA model risk approach from earlier work. The classical way to deal with model risk is to reserve the differences between the valuations in reference models and in the local models used by traders. However, while traders' prices are thus corrected, their hedging strategies and their exercise decisions are still wrong, which necessitates a risk-adjusted reserve. We illustrate our approach on a stylized callable range accrual representative of huge amounts of structured products on the market. We show that a model risk reserve adjusted for the risk of wrong exercise decisions may largely exceed a basic reserve only accounting for valuation differences.

Keywords: Pricing models; callable assets; early exercise; model risk; model calibration; cross valuation adjustments (XVAs).

*Corresponding author.

1. Introduction

The 2008 global financial crisis triggered a shift from trade-specific pricing to netting-set CVA analytics. For tractability reasons, the market models used by banks for their CVA analytics are simpler than the ones that they use for individual deals. Given this coexistence of models, it is no surprise if FRTB emphasized the issue of model risk. Traditionally, banks manage model risk by reserving the difference between asset valuations in reference models and in the local models used by traders, which broadly corresponds to a reserve for recalibration valuation leakages. However, once prices are thus corrected, hedges and exercise decisions are still wrong. Hence there is residual risk and the reserve should be risk-adjusted.

In the context of structured products, Albanese *et al.* (2021b) introduced the notion of Darwinian model risk, where the Darwinian terminology refers to the embedded adverse selection of local models by traders. Namely, when a trader wants to deal a structured product with a client, the competition for clients may lead the trader to prefer a lower quality model that outputs a price more favorable to the client (*first Darwinian principle*). But the recalibration of such a model introduces alpha leakage on the asset valuation side, which thus has to be compensated on the hedging side so that the model stands a chance to be accepted by the management of the bank (*second Darwinian principle*). However, systematic gains on the hedging side of the position is a short-to-medium viewpoint: in the long run, the falsity of the trader's model is revealed under extreme market conditions in which the local model no longer calibrates, forcing a "bad" trader to a suboptimal exercise decision or a "not-so-bad" trader to switch to a higher quality model, at the cost of more or less substantial losses for the bank (*third Darwinian principle*). Risk magazine thus reported that Q4 of 2019, a \$70bn notional of range accrual had to be unwound at very large losses by the industry: cf. *Remembering the range accrual bloodbath*[a] in which banks incurred losses of "approximately \$2.5 billion" and "never fully recovered", or *How axed dividends left SocGen in a €200 million hole.*[b] Albanese *et al.* (2021b) argued that Darwinian model risk was key to such structured products crises.

The notion of hedging valuation adjustment (HVA) was introduced by Burnett (2021) and Burnett & Williams (2021) to account for dynamical hedging transaction costs into prices. As these costs are nonlinear, they cannot be assessed for individual deals, they should be computed at the hedging set level. This feature justifies considering these costs as cross valuation adjustment (XVAs), understood as costs linked to risks, such as counterparty, funding, and capital risks, which can only be assessed at the portfolio level (Crépey, 2025). On top of transaction costs, Bénézet & Crépey (2024) incorporate in the HVA the impact of model risk, accounting for recalibration valuation leakages, by setting aside as a reserve the difference

[a]https://t.ly/W9ieL.
[b]https://t.ly/rEFA5.

between (buying) prices in bad models and prices in good models, but also for the risk of explosion of the trader's strategy. Moreover, they risk-adjust the model risk reserve by a KVA component. We refer to the introduction of their paper for a more extensive discussion about the genesis of HVA and a discussion of the model risk literature. More recent works related to model risk include Silotto *et al.* (2024) and Burnett *et al.* (2025) for model risk within a XVA environment, Sauldubois & Touzi (2024) and Fan *et al.* (2025) for sensitivities of martingale optimal transport problems, in the line of Bartl *et al.* (2021), and eventually Matsumoto & Suyama (2024), Gianfreda & Scandolo (2024) and Lazar *et al.* (2024) regarding the use of risk metrics to measure model risk in various financial settings.

But Bénézet & Crépey (2024) was only focusing on European claims or portfolios. In the case of callable assets, there is also the model risk of erroneous exercise decisions, which was in fact the main motivation and focus in Albanese *et al.* (2021b). In the present paper, we extend to callable assets the HVA take on model risk of Bénézet & Crépey (2024), thus providing mathematical and quantitative foundations to Albanese *et al.* (2021b)'s pioneering intuition. As an illustration of our approach, we devise an explicit example, stylized but representative of a very popular and liquid structured product on the market, namely the callable range accrual, where Darwinian model risk can be brought to light mathematically and quantified numerically. The consideration of the bad versus not-so-bad traders allows assessing the relevance of the proposed HVA and KVA metrics in terms of their sensitivities to the specification of the setup.

1.1. *Setup*

The risk-free asset is chosen as the numéraire. We work in a probabilistic setup $(\Omega, \mathcal{A}, \mathfrak{F}, \mathbb{Q})$, where $\mathfrak{F} = (\mathfrak{F}_t)_{t\in\mathbb{T}}$ is a continuous-time filtration with $\mathbb{T} = [0, T]$ or a discrete-time one with $\mathbb{T} = 0 \ldots T$, for a finite time horizon $T > 0$ (assumed integer in discrete time), interpreted as the final maturity of the portfolio of a bank; the fininsurance probability measure $\mathbb{Q}$ is the hybrid of pricing and physical probability measures defined in Artzner *et al.* (2024, Proposition 4.1), advocated in Albanese *et al.* (2021a, Remark 2.3) for XVA computations. We assume the bank and its counterparty default-free, referring the reader to Bénézet & Crépey (2024, Sec. 5) for the addition of unhedgeable counterparty credit risk features to our setup. For an integrable semimartingale (in a càdlàg version if in continuous time, implicitly) $\mathcal{Y} = (\mathcal{Y}_t)_{t\in[0,T]}$ starting at 0, interpreted as cumulative cash flow process of some financial derivative, we define its fair value process $Y = va(\mathcal{Y})$, respectively, its fair callable (at constant recovery $R \in [0, 1]$) value process $Z = \widetilde{va}(\mathcal{Y})$ (for $\mathcal{Y}$ uniformly integrable over $\mathbb{T}$), by

$$Y_t = \mathbb{E}_t[\mathcal{Y}_T - \mathcal{Y}_t], \quad t \in \mathbb{T}, \text{ respectively,} \tag{1.1}$$

$$Z_t = \operatorname{esssup}_{\tau\in\mathcal{T}^t} \mathbb{E}_t[\mathcal{Y}_\tau - \mathcal{Y}_t + RZ_\tau], \quad t \in \mathbb{T}, \tag{1.2}$$

where $\mathbb{E}_t$ is the conditional expectation operator with respect to $\mathfrak{F}_t$ under the measure $\mathbb{Q}$, $\mathcal{T}^t$ denotes the set of the $[t, T] \cap \mathbb{T}$ valued $\mathfrak{F}$ stopping times, and R in $[0, 1]$ is a recovery rate upon call (we will quickly reduce our attention to a standard optimal stopping setup where $R = 0$). In particular, $Y + \mathcal{Y}$ is a martingale and assuming Eq. (1.2) is well posed (e.g. for $R = 0$), $Z + \mathcal{Y}$ is a supermartingale. We denote by $\mathbb{Q}_t, \mathbb{V}a\mathbb{R}_t$ and $\mathbb{ES}_t$ the $(\mathfrak{F}_t, \mathbb{Q})$ conditional probability, value-at-risk (at some given confidence level $\alpha \in (\frac{1}{2}, 1)$ which is fixed throughout the paper) and expected shortfall (in the tail conditional expectation sense of an expected loss given this loss exceeds its value at risk).

As in Bénézet & Crépey (2024), we consider a dual-model environment: on one side, a global, fair valuation model (akin to the "reference model" advocated for model risk assessment in Barrieu & Scandolo (2015)), in which European prices (respectively, prices of callable assets) are value processes as per (1.1) (respectively, callable value processes as per (1.2)) of the corresponding cash flow; on the other side, local models used by traders for handling their deals. Due to the use of a local model (even recalibrated at all times to the fair valuation one), the raw profit-and-loss process of a deal (raw in the sense of not accounting for model risk reserves) is not a $\mathbb{Q}$ martingale in the global model, not even in the case of a European deal.

Remark 1.1. The use and (re)calibration of local models plays a central role throughout this work — from the deviation of the raw profit-and-loss process from $\mathbb{Q}$ martingality, to the explosion of the local model, when calibration becomes impossible. For simplicity, we assume in this study that calibration is either perfect or impossible. This is satisfied in our numerical example, where analytic formulas for calibrated parameters and explicit explosion times of the local model are available. We do not analyze intermediate cases in which the trader would employ a poorly calibrated model, whether this is due to infrequent recalibration or approximate calibration via numerical optimizers. Considering such intermediate calibration scenarios would substantially increase theoretical and numerical complexity, while obscuring the financial interpretation.

These deviations from martingality due to the use of local models by traders deserve a risk-adjusted reserve, so that the profit-and-loss process of the bank adjusted for the reserve becomes a submartingale in line with a remuneration of the shareholders of the bank at some hurdle rate h (e.g. 10%). We proceed in two steps. First, an HVA reserve computable deal by deal (or at the hedging set level if it also accounts for dynamic hedging transaction costs the way addressed in Bénézet & Crépey (2024, Sec. 3)) makes the profit-and-loss a martingale at each deal (or hedging set) level. Second, the profit-and-loss of the bank is risk-adjusted by a KVA reserve, computed at the level of the balance-sheet of the bank as whole.

Remark 1.2. A related concern is about the possibility of double-counting risk when incorporating multiple valuation adjustments. In the present framework, the risk of double-counting among valuation adjustments is carefully mitigated by the distinct roles and scopes of each adjustment. The HVA specifically quantifies and centers the expected losses arising from model risk and suboptimal hedging strategies or exercise decisions, adjusting the valuation distribution to reflect these realized or anticipated losses. In contrast, the KVA is designed to cover the tail risk by providing compensation for extreme loss scenarios that exceed expected losses. Because the HVA and KVA address different aspects of risk — expected losses versus tail losses — and operate on separate layers of the risk distribution, they are complementary rather than overlapping. Thus, the framework avoids double-counting by ensuring that the HVA corrects for average model-related losses, while the KVA addresses capital costs for extreme risk, leading to a consistent and additive valuation adjustment structure.

1.2. *Outline and notation*

Section 2 is dedicated to the definition and theoretical study of these HVA and KVA in the case of (derivatives portfolios including) callable claims. We detail two particular cases, associated to the aforementioned bad and a not-so-bad trader. We then illustrate the theory by computing the corresponding metrics for a stylized but "typical" structured product (callable range accrual) in a discrete time setting. Section 3 introduces the product and specifies associated global and local models. Section 4 provides a detailed numerical analysis and interpretation of the reserves, decomposing them across factors such as valuation switch, suboptimal exercise, and incorrect hedging strategy. It turns out that the risk-adjusted reserve for model risk can be significantly more substantial than the mere valuation difference between models. This highlights the importance of accounting for the misspecification of hedging and exercise strategies in model risk reserves. Section 5 concludes.

We write $\mathbf{1}_A$ and $\mathbb{1}_A$ for the indicator of a deterministic or random set A, and $x^{\pm} = \pm x \mathbf{1}_{\pm x > 0}$ for any real x. We denote by $\boldsymbol{\delta}_{\theta}$, a Dirac measure at time θ, and by X^{θ}, a process X stopped at time θ.

2. HVA for Callables: Abstract Framework

This section introduces our dual-model setup, similarly as in Bénézet & Crépey (2024). We refer to the later sections of the paper for a detailed example with illustrative numerics.

We assume that the bank buys a callable structured product from a client. The trader of the bank uses a local model to price and statically hedge the deal. At least, this holds up to a positive stopping time $\tau_s \in \mathcal{T}^0$, called model switch time, at which, if the deal has not yet been terminated, the traders starts using the global model. In this scenario, new hedging ratios are computed and the static hedging strategy is rebalanced at τ_s. As in Bénézet & Crépey (2024), one could add to the trader's strategy a dynamic hedging component. Our focus in this work is on the static hedging side, hence we refrain from doing so, to alleviate the notation. Denoting by $\tau_e \in \mathcal{T}^0$ an exercise time chosen by the trader for the asset, the raw profit-and-loss process of the trader is thus given, for $t < \tau_e$ in $\mathbb{T}$, by

$$\begin{aligned} pnl_t &= \mathcal{Q}_t + q_t \mathbb{1}_{\{t<\tau_s\}} + Q_t \mathbb{1}_{\{t\geq\tau_s\}} - q_0 \\ &\quad - (\mathcal{P}^{\rm loc}_{t\wedge\tau_s} + p^{\rm loc}_{t\wedge\tau_s} - p^{\rm loc}_0) - \mathbb{1}_{\{t\geq\tau_s\}}(\mathcal{P}^{\rm fair}_t - \mathcal{P}^{\rm fair}_{\tau_s} + P^{\rm fair}_t - P^{\rm fair}_{\tau_s}), \end{aligned} \quad (2.1)$$

where:

- $\mathcal{Q}$ denotes the cumulative cash flow process promised by the client to the bank through the deal, while $\mathcal{P}^{\rm loc}$ (respectively, $\mathcal{P}^{\rm fair}$) denotes the cumulative cash flow process promised by the bank to the hedging markets through a static hedging component constructed at time $t = 0$ (respectively, constructed at time $t = \tau_s$); the processes $\mathcal{Q}$, $\mathcal{P}^{\rm loc}$ and $\mathcal{P}^{\rm fair}$ are assumed to be integrable (uniformly over $\mathbb{T}$, regarding $\mathcal{Q}$), optional, and stopped at T;
- q (respectively, $p^{\rm loc}$) is the price of the deal (respectively, of its time-0 static hedging component, assumed European), computed by the trader of the bank in the setup of a local model used for pricing and hedging the deal before the stopping time τ_s;
- $Q = \widetilde{va}(\mathcal{Q})$ (respectively, $P^{\rm fair} = va(\mathcal{P}^{\rm fair})$) is the fair callable value of the deal (respectively, the fair value of its time-τ_s static hedging component, assumed European), used by the trader of the bank from time τ_s onwards.

The formula (2.1) is similar to Bénézet & Crépey (2024, Eq. (2)),

- with Q, the fair callable value of $\mathcal{Q}$ here, instead of its fair value there,
- with, for $t \in \mathbb{T}$, the abstract quantities $\mathcal{P}_t$, p_t and P_t there specified as $\mathcal{P}_t := \mathcal{P}^{\rm loc}_{t\wedge\tau_s} + \mathbb{1}_{\{t\geq\tau_s\}}(p^{\rm loc}_{\tau_s} - P^{\rm fair}_{\tau_s} + \mathcal{P}^{\rm fair}_t - \mathcal{P}^{\rm fair}_{\tau_s})$, $p_t \mathbb{1}_{\{t<\tau_s\}} = p^{\rm loc}_t \mathbb{1}_{\{t<\tau_s\}}$, and $P_t \mathbb{1}_{\{t\geq\tau_s\}} := P^{\rm fair}_t \mathbb{1}_{\{t\geq\tau_s\}}$ here,
- and here without the dynamic hedging component there.

Moreover, in the present setup of a callable asset with recovery rate $R \in [0,1]$, the raw profit-and-loss process may additionally jump at the exercise time τ_e, by the amount (cf. Eq. (2.1))

$$\begin{aligned} pnl_{\tau_e} - pnl_{\tau_e-} &= \mathcal{Q}_{\tau_e} - \mathcal{Q}_{\tau_e-} + R\left(q_{\tau_e}\mathbb{1}_{\{\tau_e<\tau_s\}} + Q_{\tau_e}\mathbb{1}_{\{\tau_e\geq\tau_s\}}\right) \\ &\quad - q_{\tau_e-}\mathbb{1}_{\{\tau_e\leq\tau_s\}} - Q_{\tau_e-}\mathbb{1}_{\{\tau_e>\tau_s\}} \\ &\quad - \left(\mathcal{P}^{\rm loc}_{\tau_e}\mathbb{1}_{\{\tau_e<\tau_s\}} + \mathcal{P}^{\rm fair}_{\tau_e}\mathbb{1}_{\{\tau_e\geq\tau_s\}} - \mathcal{P}^{\rm loc}_{\tau_e-}\mathbb{1}_{\{\tau_e\leq\tau_s\}} - \mathcal{P}^{\rm fair}_{\tau_e-}\mathbb{1}_{\{\tau_e>\tau_s\}}\right) \\ &\quad - \left(p^{\rm loc}_{\tau_e}\mathbb{1}_{\{\tau_e<\tau_s\}} + P^{\rm fair}_{\tau_e}\mathbb{1}_{\{\tau_e\geq\tau_s\}} - p^{\rm loc}_{\tau_e-}\mathbb{1}_{\{\tau_e\leq\tau_s\}} - P^{\rm fair}_{\tau_e-}\mathbb{1}_{\{\tau_e>\tau_s\}}\right), \end{aligned} \quad (2.2)$$

where $p^{\rm loc}_{\tau_e}\mathbb{1}_{\{\tau_e<\tau_s\}} + P^{\rm fair}_{\tau_e}\mathbb{1}_{\{\tau_e\geq\tau_s\}}$ is the liquidation cash flow of the hedge, assumed liquidly tradable at all times. A recovery rate $R = 1$ on the asset upon call would mean that the asset is liquidly sold at τ_e; $R < 1$ covers the more realistic case of a structured product that is illiquid and can only be called by the bank for a fraction of its value at τ_e.

Hereafter we assume $R = 0$, i.e. the asset is callable at zero recovery. In particular, from (1.2) with $R = 0$, the classical theory of optimal stopping (see e.g. the seminal works Neveu (1975), for the discrete time case, and El Karoui (1981, Chapter II), in continuous time) indicates that, at any time $t \in \mathbb{T}$, an optimal exercise time starting from t for the problem $Q = \widetilde{va}(\mathcal{Q})$ is given by

$$\tau^t := \inf\{\mathbb{T} \ni s \geq t; Q_s = 0\} \wedge T. \tag{2.3}$$

In addition, $\mathcal{Q}+Q$ is a supermartingale, and we denote by K its drift, i.e. the unique nondecreasing integrable predictable process such that $K_0 = 0$ and $\mathcal{Q}+Q+K$ is a martingale.

Gathering (2.1) and (2.2) for $R = 0$, we obtain the following definition.

Definition 2.1. The raw profit-and-loss process of the trader is given, for all $t \in \mathbb{T}$, by

$$\begin{aligned} pnl_t = {} & \mathcal{Q}_{t\wedge\tau_e} + q_{t\wedge\tau_e}\mathbb{1}_{\{t\wedge\tau_e<\tau_s\}} + Q_{t\wedge\tau_e}\mathbb{1}_{\{t\wedge\tau_e\geq\tau_s\}} - q_0 \\ & - \left(\mathcal{P}^{\text{loc}}_{t\wedge\tau_e\wedge\tau_s} + p^{\text{loc}}_{t\wedge\tau_e\wedge\tau_s} - p^{\text{loc}}_0\right) - \left(\mathcal{P}^{\text{fair}}_{t\wedge\tau_e} - \mathcal{P}^{\text{fair}}_{\tau_s} + P^{\text{fair}}_{t\wedge\tau_e} - P^{\text{fair}}_{\tau_s}\right)\mathbb{1}_{\{t\wedge\tau_e\geq\tau_s\}} \\ & - \mathbb{1}_{\{t\geq\tau_e\}}\left(\mathbb{1}_{\{\tau_e<\tau_s\}}q_{\tau_e} + \mathbb{1}_{\{\tau_e\geq\tau_s\}}Q_{\tau_e}\right). \end{aligned} \tag{2.4}$$

Remark 2.1. (i) Because of model risk, *pnl* fails to be a martingale, as opposed to the model-risk-free version of (2.4),

$$\mathcal{Q}_{t\wedge\tau^*_e} + Q_{t\wedge\tau^*_e} - Q_0 - \left(\mathcal{P}^*_{t\wedge\tau^*_e} + P^*_{t\wedge\tau^*_e} - P^*_0\right)$$

that would result from using only the global model everywhere (assuming optimal exercise $\tau^\star_e = \tau^0$ as per (2.3) such that, in particular $Q_{\tau^*_e} = 0$). In the equation above, we use $*$ to emphasize that the hedges and exercises decisions computed within the global model would differ from the ones in (2.4), which are computed within the trader's local model at time $t = 0$. (ii) There may also be American claims puttable by the clients, as opposed to callable by the bank (with zero recovery for notation simplicity) in the paper. In the case of puttable claims, we conservatively assume that they are optimally exercised by the clients, without benefit for the bank, so that we do not need to introduce the corresponding "nonincreasing processes" that would play a role symmetrical to our nondecreasing processes K for callable claims.

We make the following natural assumption regarding the local model used by the trader.

Assumption 2.1. *For all $t \in \mathbb{T}$, on $\{t < \tau_s\}$, the local model is calibrated to the static hedging instruments' fair prices, i.e. one has*

$$p^{\text{loc}}_t = P^{\text{loc}}_t := va(\mathcal{P}^{\text{loc}})_t. \tag{2.5}$$

In addition, we assume that $q_t = q_t^t$, for some price process $(q_s^t)_{t\le s\in\mathbb{T}}$ of the asset computed in a local model calibrated at time t to the fair valuation of the hedging assets (assumed European).

Last, we assume that, for each $t \in \mathbb{T}$, on $\{t < \tau_s\}$, in the time-t calibrated local model, an optimal stopping time for the callable deal is given by

$$\theta^t := \inf\left\{s \in [t, T) \cap \mathbb{T}; q_s^t = 0\right\} \wedge T. \tag{2.6}$$

Note that q_t and p_t^{loc} are completely unspecified on $\{t \ge \tau_s\}$, but in view of (2.4) they are irrelevant on this set.

Bénézet & Crépey (2024) was restricted to European-style structured products with τ_e constrained to be identically T in (2.4), considering various static and/or dynamic hedging strategies in this setup. In the present paper, instead, we play with various stopping times τ_e reflecting optimal calls by the trader from the viewpoint of different models, also depending on the trader's ability and willingness to switch to the fair valuation model if his local model no longer calibrates.

2.1. *Hedging valuation adjustment*

The HVA is defined as a reserve imposed by the bank to the trader to cope with misvaluation model risk, so that the HVA-compensated *pnl*, $pnl - \mathrm{HVA} + \mathrm{HVA}_0$, is a martingale (see Remark 2.1(i)):

Definition 2.2. The hedging valuation adjustment (HVA) is

$$\mathrm{HVA} = -va(pnl). \tag{2.7}$$

Proposition 2.1. *Under Assumption* 2.1, *we have, for all $t \in \mathbb{T}$,*

$$\begin{aligned}\mathrm{HVA}_t &= (q_{t\wedge\tau_e} - Q_{t\wedge\tau_e})\,\mathbb{1}_{\{t\wedge\tau_e<\tau_s\}} - \mathbb{E}_t\left[(q_{\tau_e} - Q_{\tau_e})\,\mathbb{1}_{\{\tau_e<\tau_s\}}\right]\\ &\quad + (\mathcal{Q} + Q)_{t\wedge\tau_e} - \mathbb{E}_t\left[\mathcal{Q}_{\tau_e} + Q_{\tau_e}\right]\\ &\quad + \mathbb{1}_{\{t<\tau_e\}}\mathbb{E}_t\left[\mathbb{1}_{\{\tau_e<\tau_s\}}q_{\tau_e} + \mathbb{1}_{\{\tau_e\ge\tau_s\}}Q_{\tau_e}\right].\end{aligned} \tag{2.8}$$

Proof. Recall $X^{\tau_e} := X_{\cdot\wedge\tau_e}$. Let $t \in \mathbb{T}$. Under Assumption 2.1, each of the two parentheses in the second line in (2.4), hence this second line as a whole, is a zero-valued martingale. Therefore

$$\begin{aligned}va(pnl)_t &= va\left(\left(\mathcal{Q} + q\mathbb{1}_{\{\cdot<\tau_s\}} + Q\mathbb{1}_{\{\cdot\ge\tau_s\}}\right)^{\tau_e}\right)_t\\ &\quad - va\left(\mathbb{1}_{\{\cdot\ge\tau_e\}}\left(\mathbb{1}_{\{\tau_e<\tau_s\}}q_{\tau_e} + \mathbb{1}_{\{\tau_e\ge\tau_s\}}Q_{\tau_e}\right)\right)_t\\ &= va\left(\left(\mathcal{Q} + Q + K + (q - Q)\,\mathbb{1}_{\{\cdot<\tau_s\}} - K\right)^{\tau_e}\right)_t\\ &\quad - va\left(\mathbb{1}_{\{\cdot\ge\tau_e\}}\left(\mathbb{1}_{\{\tau_e<\tau_s\}}q_{\tau_e} + \mathbb{1}_{\{\tau_e\ge\tau_s\}}Q_{\tau_e}\right)\right)_t\\ &= va\left(\left((q - Q)\,\mathbb{1}_{\{\cdot<\tau_s\}}\right)^{\tau_e}\right)_t\\ &\quad - va(K^{\tau_e})_t - va\left(\mathbb{1}_{\{\cdot\ge\tau_e\}}\left(\mathbb{1}_{\{\tau_e<\tau_s\}}q_{\tau_e} + \mathbb{1}_{\{\tau_e\ge\tau_s\}}Q_{\tau_e}\right)\right)_t,\end{aligned}$$

where we also used that $\mathcal{Q}+Q+K$ (with K as introduced in the third bullet point after (2.1)) is a martingale. Thus, by (2.7),

$$\begin{aligned}\mathrm{HVA}_t &= va\left(\left((Q-q)\,\mathbb{1}_{\{\cdot<\tau_s\}}\right)^{\tau_e}\right)_t + va(K^{\tau_e})_t \\ &\quad + va\left(\mathbb{1}_{\{\cdot\geq\tau_e\}}\left(\mathbb{1}_{\{\cdot<\tau_s\}}q + \mathbb{1}_{\{\cdot\geq\tau_s\}}Q\right)\right)_t. \qquad (2.9)\end{aligned}$$

We now compute each term separately in (2.9). First we have by (1.1), since also $\tau_e \leq T$,

$$\begin{aligned} va\left(\left((Q-q)\,\mathbb{1}_{\{\cdot<\tau_s\}}\right)^{\tau_e}\right)_t &= \mathbb{E}_t\left[\left((Q-q)\,\mathbb{1}_{\{\cdot<\tau_s\}}\right)_T^{\tau_e}\right] - \left((Q-q)\,\mathbb{1}_{\{\cdot<\tau_s\}}\right)_t^{\tau_e} \\ &= (q_{t\wedge\tau_e} - Q_{t\wedge\tau_e})\,\mathbb{1}_{\{t\wedge\tau_e<\tau_s\}} - \mathbb{E}_t\left[(q_{\tau_e} - Q_{\tau_e})\,\mathbb{1}_{\{\tau_e<\tau_s\}}\right].\end{aligned}$$

Next, since $\mathcal{Q}+Q+K$ is a martingale and $T\wedge\tau_e=\tau_e$, we have

$$va(K^{\tau_e})_t = -va((\mathcal{Q}+Q)^{\tau_e})_t = (\mathcal{Q}+Q)_t^{\tau_e} - \mathbb{E}_t\left[\mathcal{Q}_{\tau_e}+Q_{\tau_e}\right].$$

Last, since $T\wedge\tau_e=\tau_e$, we compute

$$\begin{aligned} & va\left(\mathbb{1}_{\{\cdot\geq\tau_e\}}\left(\mathbb{1}_{\{\tau_e<\tau_s\}}q_{\tau_e} + \mathbb{1}_{\{\tau_e\geq\tau_s\}}Q_{\tau_e}\right)\right)_t \\ &\quad = \mathbb{E}_t\left[\mathbb{1}_{\{T\geq\tau_e\}}\left(\mathbb{1}_{\{\tau_e<\tau_s\}}q_{\tau_e} + \mathbb{1}_{\{\tau_e\geq\tau_s\}}Q_{\tau_e}\right)\right] \\ &\qquad - \mathbb{1}_{\{t\geq\tau_e\}}\left(\mathbb{1}_{\{\tau_e<\tau_s\}}q_{\tau_e} + \mathbb{1}_{\{\tau_e\geq\tau_s\}}Q_{\tau_e}\right) \\ &\quad = \mathbb{E}_t\left[\mathbb{1}_{\{\tau_e<\tau_s\}}q_{\tau_e} + \mathbb{1}_{\{\tau_e\geq\tau_s\}}Q_{\tau_e}\right] - \mathbb{1}_{\{t\geq\tau_e\}}\left(\mathbb{1}_{\{\tau_e<\tau_s\}}q_{\tau_e} + \mathbb{1}_{\{\tau_e\geq\tau_s\}}Q_{\tau_e}\right) \\ &\quad = \mathbb{E}_t\left[\mathbb{1}_{\{t<\tau_e\}}\left(\mathbb{1}_{\{\tau_e<\tau_s\}}q_{\tau_e} + \mathbb{1}_{\{\tau_e\geq\tau_s\}}Q_{\tau_e}\right)\right]. \qquad \square\end{aligned}$$

Remark 2.2. At time $t=0$, the bank pays q_0 to its client. In addition, through the first term of the HVA (2.8) valued at $t=0$, the client pays q_0-Q_0. At this stage, from the viewpoint of the bank it is as if the bank had paid Q_0 to the client, i.e. the fair valuation price is restored. So the first HVA term in (2.8) is a reserve compensating the misvaluation before the model switch. The other terms are reserves for potentially suboptimal exercise.

2.2. *Capital valuation adjustment*

While the fair valuation prices are restored via the HVA (see Remark 2.2), the hedge is still computed in the local model before the model switch, hence it can only be wrong and leave some (or even enhance) market risk, which is not taken into account through the HVA. Similarly, a reserve for suboptimal exercise is provided, but the corresponding risk is not hedged. Unhedged risk requires shareholder's capital to cover the losses $-pnl + \mathrm{HVA} - \mathrm{HVA}_0$ associated with the still wrong hedge and exercise policy. The level of capital at risk of the bank is assumed to target a certain economic capital. The bank then needs to remunerate shareholders at some hurdle rate on their capital at risk. Under a cost-of-capital approach to the management of financial derivatives, the reserve for model risk therefore needs to be risk-adjusted, in the form of a related contribution to the capital valuation adjustment (KVA) of the bank, which is the amount needed by the bank for remunerating its shareholders for their risk.

We now define the corresponding economic capital (EC) and the associated capital valuation adjustment (KVA) processes of the bank.

Definition 2.3. For all $t \in \mathbb{T}$, we set[c]

$$\begin{aligned} \mathrm{EC}_t &= \mathbb{ES}_t\big[-(pnl_{(t+1)\wedge T} - pnl_t) + \mathrm{HVA}_{(t+1)\wedge T} - \mathrm{HVA}_t\big], \\ \mathrm{KVA}_t &= h\mathbb{E}_t\left[\int_t^T e^{-h(s-t)} \max(\mathrm{KVA}_s, \mathrm{EC}_s)\boldsymbol{\mu}_t(ds)\right] \end{aligned} \tag{2.10}$$

for some positive and constant hurdle rate h (set to 10% in our numerical applications), and where $\boldsymbol{\mu}_t$ is the Lebesgue measure on $[t,T]$, if $\mathbb{T} = [0,T]$, or $\boldsymbol{\mu}_t = \sum_{s=t+1}^{T} \boldsymbol{\delta}_s$, if $\mathbb{T} = 0 \ldots T$.

This specification ensures that the bank has exactly enough KVA[d] to remunerate its shareholders at the target hurdle rate h on their capital at risk, dynamically in time.

Remark 2.3. In this work, in order to focus on the model risk associated to the use of local models and their impact on hedges and exercise strategies, we assume that the bank's portfolio is reduced to one product and its hedge. In general, the economic capital and the KVA can only be computed at the level of the bank's portfolio.

2.3. *The bad and not-so-bad traders*

In what follows we specify the above to the special cases of the bad and the not-so-bad trader introduced in Sec. 1. The two traders behave similarly from a hedging perspective, but they differ in their early exercise strategies. Hereafter, we denote by τ_e^{bad} and pnl^{bad} (respectively, τ_e^{nsb} and pnl^{nsb}) the exercise time and the raw *pnl* of the bad (respectively, not-so-bad) trader.

We assume that before τ_s the bad trader aims at exercising optimally with respect to the local model by considering the stopping time

$$\theta^\star := \inf\left\{t \in [0,\tau_s) \cap \mathbb{T};\, \theta^t = t\right\} \wedge \tau_s = \inf\{t \in [0,\tau_s) \cap \mathbb{T};\, q_t = 0\} \wedge \tau_s, \tag{2.11}$$

where θ^t is the optimal exercise time of the trader computed in the time-t calibrated model as per Assumption 2.1, and where the equality holds by definition of q after (2.5). But if the local model no longer calibrates before the asset reaches zero value in the local model, i.e. if $\theta^\star = \tau_s$, then the bad trader is unable or unwilling to reshuffle his hedge according to the prescriptions of the global model; if his position

[c] Cf. Bénézet & Crépey (2024, Sec. 4.1).
[d] At least in the continuous time setup where $\mathbb{T} = [0,T]$, cf. Crépey (2022, Remark 2.6).

is still open at τ_s, he just closes it at that time by calling the asset and unwinding the hedge under the pricing terms of the fair valuation model. Accordingly:

Definition 2.4. The exercise policy of the bad trader is given by

$$\tau_e^{\mathrm{bad}} := \theta^\star \leq \tau_s. \tag{2.12}$$

Since $\tau_e^{\mathrm{bad}} \leq \tau_s$, we obtain from (2.4) that

$$\begin{aligned} pnl_t^{\mathrm{bad}} &= \mathcal{Q}_{t\wedge\tau_e^{\mathrm{bad}}} + q_{t\wedge\tau_e^{\mathrm{bad}}}\mathbb{1}_{\{t\wedge\tau_e^{\mathrm{bad}}<\tau_s\}} + Q_{t\wedge\tau_e^{\mathrm{bad}}}\mathbb{1}_{\{t\wedge\tau_e^{\mathrm{bad}}=\tau_s\}} - q_0 \\ &\quad - \left(\mathcal{P}^{\mathrm{loc}}_{t\wedge\tau_e^{\mathrm{bad}}} + P^{\mathrm{loc}}_{t\wedge\tau_e^{\mathrm{bad}}} - P_0^{\mathrm{loc}}\right) \\ &\quad - \mathbb{1}_{\{t\geq\tau_e^{\mathrm{bad}}\}}\left(\mathbb{1}_{\{\tau_e^{\mathrm{bad}}<\tau_s\}}q_{\tau_e^{\mathrm{bad}}} + \mathbb{1}_{\{\tau_e^{\mathrm{bad}}=\tau_s\}}Q_{\tau_e^{\mathrm{bad}}}\right). \end{aligned} \tag{2.13}$$

The not-so-bad trader behaves as the bad trader before the explosion time τ_s of the local model. However, if τ_s occurs before the termination of the deal, then the not-so-bad trader switches to the global model at τ_s, after which he aims at exercising optimally according to the latter, considering the stopping time

$$\tau^\star = \tau^{\tau_s}, \tag{2.14}$$

where τ^{τ_s} is the optimal exercise time of the trader computed in the global model at time τ_s as per (2.3). As such:

Definition 2.5. The exercise time of the not-so-bad trader is given by

$$\tau_e^{\mathrm{nsb}} := \theta^\star\mathbb{1}_{\{\theta^\star<\tau_s\}} + \tau^\star\mathbb{1}_{\{\theta^\star=\tau_s\}}, \tag{2.15}$$

with $\theta^\star$ as per (2.11) and (2.6) and $\tau^\star$ as per (2.14). The raw *pnl* of the not-so-bad trader is then given by (2.4) with $\tau_e = \tau_e^{\mathrm{nsb}}$, for which both $\mathcal{P}^{\mathrm{loc}}$ and $\mathcal{P}^{\mathrm{fair}}$ are material in (2.4) (in contrast to the bad trader for which $\mathcal{P}^{\mathrm{fair}}$ is irrelevant, see (2.13)).

3. Stylized Callable Range Accrual in Discrete Time

In the sequel of the paper, we consider a stylized callable range accrual in discrete time $\mathbb{T} = 0 \ldots T$ with T positive integer, in the natural augmented filtration $\mathfrak{F} = \mathfrak{F}^N$ of a process $N = (N_l)_{0\leq l\leq T}$ such that $N_0 = 0$ and $N_{l+1} - N_l$ is an independent Poisson random variable with parameter $\gamma_l \geq 0$, for each $l \in 0 \ldots T-1$. The range accrual cumulative cash flow process is defined by

$$\mathcal{Q}_k = \sum_{l=1}^{k}\left(\mathbb{1}_{\{I_l=-1\}} - \mathbb{1}_{\{I_l=1\}}\right), \quad k \leq T, \tag{3.1}$$

where

$$I_k = I_0(-1)^{N_k} = I_l(-1)^{N_k-N_l}, \quad 0 \leq l \leq k \leq T. \tag{3.2}$$

This process I plays the role of the global model in our example.

At each time $0 \le k \le T$, the role of the local model is then played by the process $i^k = (i^k_l)_{l=k}^T$ such that, for $k \le l \le T$,

$$i^k_l = \begin{cases} 1 & \text{if } i^k_k = 1 \text{ and } n^k_l = 0, \\ -1 & \text{otherwise, i.e. if } i^k_k = -1 \text{ or } n^k_l \ge 1, \end{cases}$$

where $n^k = (n^k_l)_{k \le l \le T}$ is a process with independent increments such that $n^k_k = 0$ and $n^k_{l+1} - n^k_l$ is an independent Poisson random variable with some parameter ν^k_l, for each $l \in k \dots T-1$. The parameters $\nu^k_l, l \in k \dots T-1$, are recalibrated at each time k (as long as it is possible) to the time-k fair values $P_k(\ell)$ of the binary options with payoff $\mathbb{1}_{\{I_\ell = -1\}}$, $\ell \in k \dots T$, which will be used a static hedging assets for the claim (see Assumption 3.1 below). Note that $P_k(k) = 0$ (respectively, $P_k(k) = 1$) if $I_k = +1$ (respectively, $I_k = -1$).

Remark 3.1. (i) In the market a typical range accrual pays a reference rate to the bank whenever this rate is outside a corridor. Our event $\{I_t = 1\}$ mimics the normal situation where the underlying rate would be inside the corridor at time t, while the event $\{I_t = -1\}$ corresponds to the extreme case where the rate would be outside the corridor. We use this simple example as a proxy to investigate the features of model risk that may have been responsible for huge losses in the structured product crises mentioned in the introduction of the paper. Namely, the bank which buys the product is long the extreme event on the asset side, but also accounting for its misspecified hedge, it will end-up short the extreme event. This is the key picture that we want to capture in our setup.

(ii) In the local model, whenever the extreme event (which the bank is long of on the asset side) occurs, then it persists until maturity. Hence the local model puts more weight on the scenarios that benefit to the bank on the asset side. In particular, the premium of the asset computed in the local model will be higher than the one in the fair valuation model (cf. Fig. 2 in Sec. 4). This induces an attractive price for the client selling the asset to the bank, which is the source of "Darwinian model risk" (of adverse model selection) in Albanese *et al.* (2021b): see Sec. 1.

(iii) As we are in discrete time and that the processes I and i^k can only take two values ± 1, our setup is amenable to exact numerical evaluation, without Monte Carlo simulation or partial differential equation (PDE) approximation biases (see Secs. 3 and 4).

Hereafter in this section, we study the theoretical properties of our stylized range accrual.

3.1. *Pricing and hedging*

3.1.1. *Hedging assets and calibration of the local model*

We have the following two lemmas regarding the pricing of the binary options in the fair valuation and local models. These binary options being used as calibration and static hedging assets, we deduce as a corollary the calibration of the local model to these fair valuation prices.

Lemma 3.1. *The time-k fair valuation price of the binary option with maturity ℓ is given, for each $0 \le k \le \ell \le T$, by*

$$P_k(\ell) = \mathbb{1}_{\{I_k=1\}} \frac{1 - e^{-2\sum_{l=k}^{\ell-1} \gamma_l}}{2} + \mathbb{1}_{\{I_k=-1\}} \frac{1 + e^{-2\sum_{l=k}^{\ell-1} \gamma_l}}{2}. \tag{3.3}$$

Proof. We compute

$$\begin{aligned} P_k(\ell) &= \mathbb{E}_k\left[\mathbb{1}_{\{I_\ell=-1\}}\right] = \mathbb{E}_k\left[\mathbb{1}_{\{I_k(-1)^{N_\ell-N_k}=-1\}}\right] \\ &= \mathbb{1}_{\{I_k=1\}}\mathbb{E}_k\left[\mathbb{1}_{\{(-1)^{N_\ell-N_k}=-1\}}\right] \\ &\quad + \mathbb{1}_{\{I_k=-1\}}\mathbb{E}_k\left[\mathbb{1}_{\{(-1)^{N_\ell-N_k+1}=-1\}}\right] \\ &= \mathbb{1}_{\{I_k=1\}}\mathbb{Q}\left[N_\ell - N_k \text{ odd}\right] + \mathbb{1}_{\{I_k=-1\}}\mathbb{Q}\left[N_\ell - N_k \text{ even}\right], \end{aligned}$$

which yields (3.3). □

Lemma 3.2. *For $0 \le k \le \ell \le T$, the time-k local model price of the binary option with maturity ℓ is*

$$\mathbb{E}_k\left[\mathbb{1}_{\{i^k_\ell=-1\}}\right] = \mathbb{1}_{\{i^k_k=-1\}} + \mathbb{1}_{\{i^k_k=1\}}(1 - e^{-\sum_{l=k}^{\ell-1} \nu^k_l}).$$

Proof. We compute

$$\begin{aligned} \mathbb{E}_k\left[\mathbb{1}_{\{i^k_\ell=-1\}}\right] &= \mathbb{1}_{\{i^k_k=-1\}} + \mathbb{1}_{\{i^k_k=1\}}\mathbb{E}_k\left[\mathbb{1}_{\{n^k_\ell\ge 1\}}\right] \\ &= \mathbb{1}_{\{i^k_k=-1\}} + \mathbb{1}_{\{i^k_k=1\}}\left(1 - \mathbb{Q}\left[n^k_\ell = 0\right]\right), \end{aligned}$$

where $\mathbb{Q}\left[n^k_\ell = 0\right] = e^{-\sum_{l=k}^{\ell-1} \nu^k_l}$. □

Corollary 3.1. *Assuming $I_0 = 1$, as long as $I_k = 1$, the local model calibrates to the term structure $P_k(\cdot)$ in (3.3) via $i^k_k = I_k = 1$ and*

$$\begin{aligned} 1 - e^{-\sum_{l=k}^{\ell-1} \nu^k_l} &= P_k(\ell), \quad k < \ell, \\ \text{i.e. } \nu^k_{\ell-1} &= -\ln(1 - P_k(\ell)) - \sum_{l=k}^{\ell-2} \nu^k_l, \quad k < \ell. \end{aligned} \tag{3.4}$$

As soon as the extreme event occurs, i.e. at

$$\tau_s = \inf\{k \in 0 \ldots T;\, I_k = -1\} \wedge T, \tag{3.5}$$

the trader's local model no longer calibrates (at least if $\tau_s < T$, and note that we neither need nor use any model at time T).

3.1.2. *Range accrual and its hedging ratios*

We now compute, for each $0 \le t \le T$, the fair valuation and local prices of the range accrual. Let, for $1 \le l \le T$,

$$\begin{aligned} u_l &= \mathbb{Q}\,[N_l - N_{l-1} \text{ even}] = \frac{1}{2}(1 + e^{-2\gamma_{l-1}}), \\ v_l &= \mathbb{Q}\,[N_l - N_{l-1} \text{ odd}] = \frac{1}{2}(1 - e^{-2\gamma_{l-1}}). \end{aligned} \tag{3.6}$$

Proposition 3.1. *The fair callable value of the range accrual at time $k \le T$ is equal to*

$$Q_k = h \sum_{\ell=k+1}^{T} \Big(A_k(\ell)\, P_k(\ell) - B_k(\ell)\big(1 - P_k(\ell)\big)\Big), \tag{3.7}$$

with

$$A_k(\ell) = \frac{\mathbb{E}_k\left[\mathbb{1}_{\{I_\ell=-1\}}\mathbb{1}_{\{\ell\le\tau^k\}}\right]}{P_k(\ell)}, \quad B_k(\ell) = \frac{\mathbb{E}_k\left[\mathbb{1}_{\{I_\ell=1\}}\mathbb{1}_{\{\ell\le\tau^k\}}\right]}{1 - P_k(\ell)}, \quad k \le \ell \le T, \tag{3.8}$$

where τ^k is the optimal exercise time computed at time k as per (2.3).

The process Q in (3.7) *can be represented as $Q_k = Q(k, I_k)$, for the pricing function $Q : \{0, \dots, T\} \times \{1, -1\} \to \mathbb{R}$ such that*

$$\begin{aligned} &Q(T, \mp 1) = 0 \text{ and for } 0 \le k < T, \\ &Q(k, -1) = e^{-2\gamma_k} + v_{k+1}Q(k+1, 1) + u_{k+1}Q(k+1, -1) > 0, \\ &Q(k, 1) = \max(0, -e^{-2\gamma_k} + u_{k+1}Q(k+1, 1) + v_{k+1}Q(k+1, -1)). \end{aligned} \tag{3.9}$$

Proof. We compute, with τ^k and $A_k(\ell)$, $B_k(\ell)$ as introduced,

$$\begin{aligned} Q_k &= \operatorname{esssup}_{\tau\in\mathcal{T}^k} \mathbb{E}_k\left[\sum_{\ell=k+1}^{T} \left(\mathbb{1}_{\{I_\ell=-1\}} - \mathbb{1}_{\{I_\ell=1\}}\right)\mathbb{1}_{\{t_\ell\le\tau\}}\right] \\ &= \sum_{\ell=k+1}^{T} \mathbb{E}_k\left[\left(\mathbb{1}_{\{I_\ell=-1\}} - \mathbb{1}_{\{I_\ell=1\}}\right)\mathbb{1}_{\{t_\ell\le\tau^k\}}\right] \\ &= \sum_{\ell=k+1}^{T} \Big(A_k(\ell)P_k(\ell) - B_k(\ell)\big(1 - P_k(\ell)\big)\Big), \end{aligned} \tag{3.10}$$

which proves (3.7).

Moreover, by the Markov property of I, the process Q can be represented as $Q_k = Q(k, I_k)$, where the function $Q(\cdot,\cdot)$ satisfies the backward dynamic programming equations $Q(T, I_T) = 0$ and for $0 \le k < T$,

$$\begin{aligned} Q(k, I_k) = \max(0, &\,\mathbb{Q}_k\,[I_{k+1} = 1]\,(-1 + Q(k+1, 1)) \\ &+ \mathbb{Q}_k\,[I_{k+1} = -1]\,(1 + Q(k+1, -1))), \end{aligned}$$

i.e. $Q(T, \mp 1) = 0$ and for $0 \le k < T$,

$$Q(k,-1) = \max\left(0, v_{k+1}\left(-1 + Q(k+1,1)\right) + u_{k+1}\left(1 + Q(k+1,-1)\right)\right),$$
$$Q(k,1) = \max\left(0, u_{k+1}\left(-1 + Q(k+1,1)\right) + v_{k+1}\left(1 + Q(k+1,-1)\right)\right). \qquad \square$$

We have the following similar statement regarding the pricing of the claim in the time-k calibrated local model, recall Assumption 2.1. The proof is similar and thus omitted.

Proposition 3.2. *For each $0 \le k \le T$, the callable price of the range accrual in the local model is equal to*

$$q_k = \sum_{\ell=k+1}^{T} \Big(a_k(\ell) P_k(\ell) - b_k(\ell)\big(1 - P_k(\ell)\big)\Big), \tag{3.11}$$

with

$$a_k(\ell) = \frac{\mathbb{E}_k\left[\mathbb{1}_{\{i_\ell^k=-1\}}\mathbb{1}_{\{\ell\le\theta^k\}}\right]}{P_k(\ell)} \quad \textit{and} \quad b_k(\ell) = \frac{\mathbb{E}_k\left[\mathbb{1}_{\{i_\ell^k=1\}}\mathbb{1}_{\{\ell\le\theta^k\}}\right]}{1 - P_k(\ell)}, \quad k \le \ell \le T, \tag{3.12}$$

where $\theta^k = \inf\{\ell \ge k;\, q_\ell^k = 0\}$, see (2.6), is an optimal stopping rule in the time-k calibrated local model.

The process q in (3.11) can be represented as $q_k = q^k(k, i_k^k) = q^k(k, I_k)$, for the pricing functions $q^k : \{k, \ldots, T\} \times \{1, -1\} \to \mathbb{Q}$ defined, for each $0 \le k \le T$, by

$$q^k(T, \mp 1) = 0 \textit{ and for } k \le l < T,$$
$$q^k(l, -1) = T - l,$$
$$q^k(l, 1) = \max\Big(0, e^{-\nu_l^k}\left(-1 + q^k(l+1,1)\right) + \Big(1 - e^{-\nu_l^k}\Big)\left(1 + q^k(l+1,-1)\right)\Big). \tag{3.13}$$

In view of (3.7)–(3.8), at any time k, a natural static hedging strategy from the global model perspective, dubbed fair hedge below, is to sell (respectively, buy), for each $k < \ell \le T$, an amount $A_k(\ell)$ (respectively, $B_k(\ell)$) of binary options with payoff $\mathbb{1}_{\{I_\ell=-1\}}$ (respectively, $\mathbb{1}_{\{I_\ell=1\}}$). This would in fact statically replicate the range accrual if it was not for its callability (the noncallable version of the range accrual is nothing but the collection of the binaries).

Likewise, in view of (3.11)–(3.12), at time k, a natural static hedging strategy from the local model perspective, dubbed local hedge below, is to sell (respectively, buy), for each $k < \ell \le T$, an amount $a_k(\ell)$ (respectively, $b_k(\ell)$) of binary options with payoff $\mathbb{1}_{\{I_\ell=-1\}}$ (respectively, $\mathbb{1}_{\{I_\ell=1\}}$).

Accordingly:

Assumption 3.1. (i) *At time $k = 0$, both traders implement the local static hedge*

$$\mathcal{P}_k^{\text{loc}} = \sum_{\ell=1}^{k} \left(a_0(\ell)\mathbb{1}_{\{I_\ell=-1\}} - b_0(\ell)\mathbb{1}_{\{I_\ell=1\}}\right), \quad k \geq 0. \tag{3.14}$$

(ii) *At the model switch time $k = \tau_s$, the bad trader unwinds its position (under the conditions prescribed by the global model), while (if $\tau_s < \tau_e^{\text{nsb}}$) the not-so-bad trader switches to the fair static hedge such that*

$$\mathcal{P}_k^{\text{fair}} = \sum_{\ell=\tau_s+1}^{k} (A_{\tau_s}(\ell)\mathbb{1}_{\{I_\ell=-1\}} - B_{\tau_s}(\ell)\mathbb{1}_{\{I_\ell=1\}}), \quad k \geq 0. \tag{3.15}$$

Remark 3.2. $\mathcal{P}^{\text{loc}}$ is fairly valued, for $k \geq 0$, as

$$P_k^{\text{loc}} = \sum_{\ell=k+1}^{T} (a_0(\ell)P_k(\ell) - b_0(\ell)(1 - P_k(\ell))), \tag{3.16}$$

with $P_k(\ell)$ as in (3.3); in particular, (3.11) and (3.16) yield that $P_0^{\text{loc}} = q_0$, which also reads $p_0^{\text{loc}} = q_0$, by Assumption 2.1. $\mathcal{P}^{\text{fair}}$ is fairly valued, for $k \geq 0$ and on $\{k \geq \tau_s\}$, as

$$P_k^{\text{fair}} = \sum_{\ell=k+1}^{T} (A_{\tau_s}(\ell)P_k(\ell) - B_{\tau_s}(\ell)(1 - P_k(\ell))). \tag{3.17}$$

The following lemma allows computing the static hedging ratios $a_0(\ell)$ and $b_0(\ell)$ for all $0 < \ell \leq T$.

Lemma 3.3. *Let $\underline{\theta}^0 := \inf\{0 \leq l \leq T;\ q^0(l, 1) = 0\} \wedge T$.*

(i) *We have $\underline{\theta}^0 \leq \theta^0 = \inf\{l \geq 0;\ q_l^0 = 0\}$ (see Proposition 3.2 and (2.6)).*
(ii) *For all $0 \leq \ell \leq \underline{\theta}^0$, one has $a_0(\ell) = b_0(\ell) = 1$.*
(iii) *For all $\underline{\theta}^0 < \ell \leq T$, one has $b_0(\ell) = 0$.*
(iv) *For all $\underline{\theta}^0 < \ell \leq T$, one has $a_0(\ell) = \frac{P_0(\ell_{\max})}{P_0(\ell)}$.*

Proof. (i) Notice that $q^0(\ell, 1) \neq 0$ for $\ell < \underline{\theta}^0$, by definition of $\underline{\theta}^0$, and $q^0(\ell, -1) = T - \ell > 0$ as $\ell < T$, see Proposition 3.2. Hence $q_\ell^0 \neq 0$ for all $0 \leq \ell < \underline{\theta}^0$, implying that $\theta^0 \geq \underline{\theta}^0$.

(ii) For $0 \leq \ell \leq \underline{\theta}^0$, (i) implies $\ell \leq \underline{\theta}^0 \leq \theta^0$, hence (3.12) yields

$$a_0(\ell) = \frac{\mathbb{E}\left[\mathbb{1}_{\{i_\ell^0=-1\}}\right]}{P_0(\ell)} \text{ and } b_0(\ell) = \frac{\mathbb{E}\left[\mathbb{1}_{\{i_\ell^0=1\}}\right]}{1 - P_0(\ell)},$$

where both quantities are equal to 1 as, by assumption, the time-0 local model is calibrated to the binary option prices (which precisely means that $\mathbb{E}\left[\mathbb{1}_{\{i_\ell^0=-1\}}\right] = P_0(\ell)$ holds for all $0 \leq \ell \leq T$).

(iii) Let $\underline{\theta}^0 < \ell \le T$. We show that $\{i^0_\ell = 1\} \cap \{\ell \le \theta^0\} = \emptyset$, which implies by (3.12) that $b_0(\ell) = 0$. If $i^0_\ell = 1$, then $i^0_0 = \cdots = i^0_\ell = 1$ (as -1 is an absorbing state in the local models). In particular, since $\underline{\theta}^0 < \ell$, $q^0(\underline{\theta}^0, i^0_{\underline{\theta}^0}) = q^0(\underline{\theta}^0, 1) = 0$ by definition of $\underline{\theta}^0$, meaning that $\theta^0 \le \underline{\theta}^0 < \ell$. This proves, as required, that $\{i^0_\ell = 1\} \cap \{\ell \le \theta^0\} = \emptyset$.

(iv) We last show, for $\underline{\theta}^0 < \ell \le T$, that $\{i^0_\ell = -1\} \cap \{\theta^0 \ge \ell\} = \{i^0_{\underline{\theta}^0} = -1\}$, which implies by (3.12) that $a_0(\ell) = \frac{\mathbb{Q}(i^0_{\underline{\theta}^0} = -1)}{P_0(\ell)}$, and the proof is concluded by invoking that $\mathbb{Q}(i^0_{\underline{\theta}^0} = -1) = P_0(\underline{\theta}^0)$ as the time-0 local model is calibrated to the binary options prices.

First, if $i^0_{\underline{\theta}^0} = -1$, then, for all $\underline{\theta}^0 \le k \le T$, $i^0_k = -1$ and $q^0_k = q^0(k, i^0_k) = q^0(k, -1) = T - k$ as -1 is an absorbing state in the local model. In particular, we have $i^0_\ell = -1$. In addition, we proved $\theta^0 \ge \underline{\theta}^0$, which implies $q^0_k \ne 0$ for all $k < \underline{\theta}^0$. Besides, $q^0_k = T - k > 0$ for all $\underline{\theta}^0 \le k < T$. In conclusion, $q^0_k \ne 0$ for all $0 \le k < T$, hence $\theta^0 = T \ge \ell$. This proves $\{i^0_{\underline{\theta}^0} = -1\} \subset \{i^0_\ell = -1\} \cap \{\theta^0 \ge \ell\}$.

Conversely, if $i^0_\ell = -1$ and $\theta^0 \ge \ell$, since by assumption $\underline{\theta}^0 < \ell$ also holds throughout this part (iv) of the proof, therefore $\underline{\theta}^0 < \ell \le \theta^0$, hence $q^0_{\underline{\theta}^0} > 0$. Since $0 < q^0_{\underline{\theta}^0} \in \{q^0(\underline{\theta}^0, 1), q^0(\underline{\theta}^0, -1)\}$ and $q^0(\underline{\theta}^0, 1) = 0$ by definition, one necessarily has $q^0_{\underline{\theta}^0} = q^0(\underline{\theta}^0, -1)$ and hence $i^0_{\underline{\theta}^0} = -1$. □

Remark 3.3. To obtain such simple formulas for $a_0(\ell)$ and $b_0(\ell)$, $0 \le \ell \le T$, we heavily make use of the fact that, in the time-0 calibrated local model, the state -1 is absorbing, which implies that $q^0(\ell, -1) = T - \ell$. This is not valid in the fair valuation model, preventing us from providing simple formulas for the hedging ratios $A_0(\ell)$ and $B_0(\ell)$. However, these can still be computed exactly, the way explained in Sec. 3.3 below (see in particular Lemma 3.7).

Hereafter, whenever a random variable ξ is constant on an event A, with a slight abuse of notation, we denote its value on A by $\xi(A)$.

3.2. *Bad trader's XVAs*

In this section, we study how to compute the various stochastic processes introduced in Sec. 2 regarding a bad trader of Definition 2.4, buying the range accrual studied in Sec. 3.1 and statically hedging it as postulated in Assumption 3.1.

Since the bad trader calls back the asset no later than the model switch time τ_s (see (2.12)), by (3.5), the only relevant events in his case are the following partition of Ω:

$$\begin{aligned} &\Omega_{T+1} = \{I_0 = 1, \dots, I_T = 1\} \text{ and for } 1 \le l \le T, \\ &\Omega_l = \{I_0 = 1, \dots, I_{l-1} = 1, I_l = -1\}, \end{aligned} \tag{3.18}$$

where Ω_{T+1} corresponds to the extreme event never occurring on $0\dots T$, while, for $k < l \le T$, Ω_l corresponds to the extreme event first occurring at time l (assuming $I_0 = 1$). Note that Ω_l is $\mathfrak{F}_l$ measurable, for each $1 \le l \le T$, while Ω_{T+1} is $\mathfrak{F}_T$ measurable. For $l \le T+1$ and $k \le l \wedge T$, $I_k(\Omega_l)$ is obviously given by $I_k(\Omega_l) = \mathbf{1}_{k<l} - \mathbf{1}_{k=l}$. The stopping times τ_s and τ_e^{bad} are also constant on each Ω_l, $1 \le l \le T+1$. Namely, (3.5) and (2.12) imply, for all $1 \le l \le T+1$:

$$\begin{aligned} &\tau_s(\Omega_l) = \inf\{k; I_k = -1\} \wedge T = l \wedge T, \\ &\tau_e^{\mathrm{bad}}(\Omega_l) = \inf\{k < l \wedge T;\, q^k(k,1) = 0\} \wedge (l \wedge T), \end{aligned} \tag{3.19}$$

which can be determined from the $q^k(k,1)$, $1 \le k \le T$, computed via (3.13). Moreover, with the notations (3.6) at hand, as proved in Sec. A.1.

Lemma 3.4. *For every $k \le T$ and $1 \le l \le T+1$, the $\mathfrak{F}_k$ conditional probabilities of the partitioning events Ω_λ, $1 \le \lambda \le T+1$, are constant on each Ω_l, where they are worth*

$$\mathbb{Q}_k\left[\Omega_\lambda\right](\Omega_l) = \mathbf{1}_{k\ge\lambda}\mathbf{1}_{l=\lambda} + \mathbf{1}_{k<\lambda}\mathbf{1}_{l>k}\left(\prod_{m=k+1}^{\lambda-1} u_m\right) v_\lambda,$$

$$1 \le \lambda \le T \quad \textit{and}$$

$$\mathbb{Q}_k\left[\Omega_{T+1}\right](\Omega_l) = \mathbf{1}_{l>k} \prod_{m=k+1}^{T} u_m. \tag{3.20}$$

Since the market is represented by the process I and the processes relative to the bad trader are all stopped at τ_e^{bad}, the corresponding study boils down to understanding computations relative to $\mathfrak{F}_{\tau_e^{\mathrm{bad}}} \cap \sigma(I_k, k \le T)$ measurable random variables. Now, for such random variable, the following properties are proved in Sec. A.1.

Lemma 3.5. (i) *Let ξ be an $\mathfrak{F}_{\tau_e^{\mathrm{bad}}} \cap \sigma(I_k, k \le T)$ measurable random variable. Then, ξ is constant on each $\Omega_l, 1 \le l \le T+1$.*

(ii) *Let ξ be a random variable constant on each Ω_l, $1 \le l \le T+1$. Then, for each $0 \le k \le T$ and $1 \le l \le T+1$, $\mathbb{E}_k[\xi]$, $\mathbb{V}a\mathbb{R}_k(\xi)$ and $\mathbb{ES}_k(\xi)$ are constant on Ω_l; in particular,*

$$\mathbb{E}_k\left[\xi\right](\Omega_l) = \sum_{\lambda=1}^{T+1} \xi(\Omega_\lambda)\mathbb{Q}_k\left[\Omega_\lambda\right](\Omega_l). \tag{3.21}$$

If in addition $l \le k$, then

$$\mathbb{E}_k\left[\xi\right](\Omega_l) = \mathbb{V}a\mathbb{R}_k(\xi)(\Omega_l) = \mathbb{ES}_k(\xi)(\Omega_l) = \xi(\Omega_l). \tag{3.22}$$

We now apply the above lemmas to the random variables associated with the bad trader.

Proposition 3.3. *Let Assumption* 3.1 *be in force. Let* $k \leq T$.

(i) *We have*

$$\begin{aligned} pnl_k^{\text{bad}} &= \mathcal{Q}_{k\wedge\tau_e^{\text{bad}}} + q_{k\wedge\tau_e^{\text{bad}}}\mathbb{1}_{\{k\wedge\tau_e^{\text{bad}}<\tau_s\}} + Q_{k\wedge\tau_e^{\text{bad}}}\mathbb{1}_{\{k\wedge\tau_e^{\text{bad}}=\tau_s\}} \\ &\quad - (\mathcal{P}^{\text{loc}}_{k\wedge\tau_e^{\text{bad}}} + P^{\text{loc}}_{k\wedge\tau_e^{\text{bad}}}) - \mathbb{1}_{\{k\geq\tau_e^{\text{bad}}\}}\mathbb{1}_{\{\tau_e^{\text{bad}}=\tau_s\}}Q_{\tau_e^{\text{bad}}}, \\ \text{HVA}_k^{\text{bad}} &= \underbrace{(q_{k\wedge\tau_e^{\text{bad}}} - Q_{k\wedge\tau_e^{\text{bad}}})\mathbb{1}_{\{k\wedge\tau_e^{\text{bad}}<\tau_s\}}}_{=:U_k^{\text{bad}}} + \underbrace{\mathbb{E}_k\left[Q_{\tau_e^{\text{bad}}}\mathbb{1}_{\{\tau_e^{\text{bad}}<\tau_s\}}\right]}_{=:V_k^{\text{bad}}} \\ &\quad + \mathbb{1}_{\{k<\tau_e^{\text{bad}}\}}\underbrace{\mathbb{E}_k\left[\mathbb{1}_{\{\tau_e^{\text{bad}}=\tau_s\}}Q_{\tau_e^{\text{bad}}}\right]}_{=:W_k^{\text{bad}}} \\ &\quad + \underbrace{\mathcal{Q}_{k\wedge\tau_e^{\text{bad}}} + Q_{k\wedge\tau_e^{\text{bad}}} - \mathbb{E}_k\left[\mathcal{Q}_{\tau_e^{\text{bad}}} + Q_{\tau_e^{\text{bad}}}\right]}_{va(K^{\tau_e^{\text{bad}}})_k}. \end{aligned} \tag{3.23}$$

(ii) *The random variables* pnl_k^{bad} *and* $\text{HVA}_k^{\text{bad}}$ *are constant on each of the* Ω_l, *where their values can be computed using Propositions* 3.1 *and* 3.2 *and Lemmas* 3.4 *and* 3.5.

(iii) EC_k^{bad}, *as defined in* (2.10) *specified to the bad trader dealing the range accrual, is constant on each of the* Ω_l, *with* $\text{EC}_k^{\text{bad}}(\Omega_l) = 0$ *for* $l \leq k$ *and a constant independent of* l, *denoted by* $\text{EC}^{\text{bad}}(k)$ *and also computable by Propositions* 3.1 *and* 3.2 *and Lemmas* 3.4 *and* 3.5, *for* $l > k$.

(iv) $\text{KVA}_k^{\text{bad}}$, *as defined in* (2.10) *specified to the bad trader dealing the range accrual, is constant on each* Ω_l, $1 \leq l \leq T+1$. *In particular, we have* $\text{KVA}_k^{\text{bad}}(\Omega_l) = 0, 1 \leq l \leq k \leq T$ *and*

$$\text{KVA}_0^{\text{bad}} = h\sum_{k=0}^{T-1} e^{-hk}\sum_{\lambda=k+1}^{T+1}\max(\text{EC}^{\text{bad}}(k), \text{KVA}_k^{\text{bad}}(\Omega_\lambda))\mathbb{Q}[\Omega_\lambda]. \tag{3.24}$$

Proof. We fix $0 \leq k \leq T$.

(i) The equations for pnl_k^{bad} and $\text{HVA}_k^{\text{bad}}$ follow from (2.13) and (2.8), recalling that $P_0^{\text{loc}} = q_0$ (see after Assumption 3.1) and $\mathbb{1}_{\{\tau_e^{\text{bad}}<\tau_s\}}q_{\tau_e^{\text{bad}}} = 0$ (by (2.12) and (2.11)).

(ii) For each $0 \leq k \leq T$, the random variables

$$\begin{aligned} &pnl_k^{\text{bad}}, U_k^{\text{bad}}, \mathbb{1}_{\{\tau_e^{\text{bad}}<\tau_s\}}Q_{\tau_e^{\text{bad}}}, \\ &\mathbb{1}_{\{k<\tau_e^{\text{bad}}\}}\mathbb{1}_{\{\tau_e^{\text{bad}}\geq\tau_s\}}Q_{\tau_e^{\text{bad}}}, \quad \text{and} \\ &\mathcal{Q}_{k\wedge\tau_e^{\text{bad}}} + Q_{k\wedge\tau_e^{\text{bad}}} - (\mathcal{Q}_{\tau_e^{\text{bad}}} + Q_{\tau_e^{\text{bad}}}) \end{aligned} \tag{3.25}$$

are obviously $\mathfrak{F}_{\tau_e^{\text{bad}}}$ measurable. From (3.19), $\tau_s = \inf\{k; I_k = -1\}$ is $\sigma(I_k, k \leq T)$ measurable, and so is $\tau_e^{\text{bad}} = \inf\{k; q^k(k,1) = 0\} \wedge \tau_s$, as $\inf\{k; q^k(k,1) = 0\}$ is deterministic. By definition (3.1) and (3.14), the processes $\mathcal{Q}$ and

$\mathcal{P}^{\text{bad}}$ are $\sigma(I_k, k \leq T)$ adapted. By Propositions 3.1 and 3.2, the processes $q = (q^k(k, I_k))_{0\leq k\leq T}$ and $Q = (Q^k(k, I_k))_{0\leq k\leq T}$ are also $\sigma(I_k, k \leq T)$ adapted. So is also P^{bad}, by (3.16) and (3.3). Hence all the random variables in (3.25) are $\mathfrak{F}_{\tau_e^{\text{bad}}} \cap \sigma(I_k, k \leq T)$ measurable. By Lemma 3.4(i), they are therefore constant on each of the Ω_l, $1 \leq l \leq T+1$. By Lemma 3.4(ii), this then implies that $V_k^{\text{bad}} = \mathbb{E}_k\left[\mathbb{1}_{\{\tau_e^{\text{bad}}<\tau_s\}} Q_{\tau_e^{\text{bad}}}\right]$, $\mathbb{1}_{\{k<\tau_e^{\text{bad}}\}} W_k = \mathbb{E}_k\left[\mathbb{1}_{\{k<\tau_e^{\text{bad}}=\tau_s\}} Q_{\tau_e^{\text{bad}}}\right]$ and $va(K^{\tau_e^{\text{bad}}})_k = \mathbb{E}_k\left[\mathcal{Q}_{k\wedge\tau_e^{\text{bad}}} + Q_{k\wedge\tau_e^{\text{bad}}} - (\mathcal{Q}_{\tau_e^{\text{bad}}} + Q_{\tau_e^{\text{bad}}})\right]$ are constant on each Ω_l, $1 \leq l \leq T+1$, and so is in turn $\text{HVA}_k^{\text{bad}}$.

(iii) EC_k^{bad} is, by (2.10), the $\mathfrak{F}_k$ conditional expected shortfall of a random variable which is, by (ii), constant on each Ω_l, $1 \leq l \leq T+1$. By Lemma 3.4(ii), EC_k^{bad} is also constant on each Ω_l. Moreover, if $l \leq k$, (3.22) shows that $\text{EC}_k^{\text{bad}}(\Omega_l) = -(pnl^{\text{bad}}_{(k+1)\wedge T}(\Omega_l) - pnl^{\text{bad}}_k(\Omega_l)) + \text{HVA}^{\text{bad}}_{(k+1)\wedge T}(\Omega_l) - \text{HVA}^{\text{bad}}_k(\Omega_l)$. But this is equal to 0 as the processes pnl^{bad} and HVA^{bad} are stopped at τ_e^{bad} (see (3.23)), which is $\leq l$ on Ω_l (see (3.19)). Moreover, the first line of (3.20) shows that $\mathbb{Q}_k[\Omega_\lambda](\Omega_l)$ is equal to 0 for $l \leq k$ and does not depend on l for $l > k$, which implies the last statement regarding EC.

(iv) By backward induction on k, $\text{KVA}_k^{\text{bad}}$ is constant on each Ω_l, $1 \leq l \leq T+1$. In fact, $\text{KVA}_T^{\text{bad}} = 0$, while assuming the induction hypothesis at rank $k+1$ yields by (2.10) that $\text{KVA}_k^{\text{bad}}$ is the $\mathfrak{F}_k$ conditional expectation of a random variable which is, by (iii), constant on each Ω_l, $1 \leq l \leq T+1$. Hence, by Lemma 3.5(ii), $\text{KVA}_k^{\text{bad}}$ is also constant on each Ω_l, $1 \leq l \leq T+1$. In addition, we have, by (2.10) (in discrete time), for $1 \leq l \leq k \leq T$,

$$\begin{aligned}
&\text{KVA}_k^{\text{bad}}(\Omega_l)\\
&\quad = h\mathbb{E}_k\left[\sum_{t=k+1}^{T} e^{-h(t-k)} \max(\text{KVA}_t^{\text{bad}}, \text{EC}_t^{\text{bad}})\right](\Omega_l)\\
&\quad = h\sum_{\lambda=1}^{T+1}\left(\sum_{t=k+1}^{T} e^{-h(t-k)} \max(\text{KVA}_t^{\text{bad}}(\Omega_\lambda), \text{EC}_t^{\text{bad}}(\Omega_\lambda))\right)\mathbb{Q}_k[\Omega_\lambda](\Omega_l),
\end{aligned}$$

by (3.21). By (3.20), since $l \leq k$, $\mathbb{Q}_k[\Omega_\lambda](\Omega_l) = \mathbf{1}_{\lambda=l}$. Hence

$$\text{KVA}_k^{\text{bad}}(\Omega_l) = h\sum_{t=k+1}^{T} e^{-h(t-k)} \max(\text{KVA}_t^{\text{bad}}(\Omega_l), \text{EC}_t^{\text{bad}}(\Omega_l)).$$

By (iii), we have $\text{EC}_t^{\text{bad}}(\Omega_l) = 0$ as $l \leq k < t$, hence

$$\text{KVA}_k^{\text{bad}}(\Omega_l) = h\sum_{t=k+1}^{T} e^{-h(t-k)}\big(\text{KVA}_t^{\text{bad}}(\Omega_l)\big)^{+}$$

and a straightforward backward induction in k, starting from $\text{KVA}_T^{\text{bad}}(\Omega_l) = 0$, shows that $\text{KVA}_k^{\text{bad}}(\Omega_l) = 0$ for $k \geq l$.

Besides, (2.8) yields

$$\begin{aligned}\mathrm{KVA}_0^{\mathrm{bad}} &= h\mathbb{E}\left[\sum_{l=1}^{T} e^{-hl}\max(\mathrm{KVA}_l^{\mathrm{bad}}, \mathrm{EC}_l^{\mathrm{bad}})\right]\\ &= h\sum_{l=1}^{T} e^{-hl}\sum_{\lambda=1}^{T+1}\max(\mathrm{KVA}_l^{\mathrm{bad}}(\Omega_\lambda), \mathrm{EC}_l^{\mathrm{bad}}(\Omega_\lambda))\mathbb{Q}[\Omega_\lambda]\end{aligned}$$

and since $\mathrm{KVA}_l^{\mathrm{bad}}(\Omega_\lambda) = \mathrm{EC}_l^{\mathrm{bad}}(\Omega_\lambda) = 0$ for $\lambda \le l$, we obtain (3.24). □

3.3. *Not-so-bad trader's XVAs*

We now perform the computations regarding the not-so-bad trader of Definition 2.5, buying the range accrual and statically hedging it along Assumption 3.1.

To ease the study, we make the following assumption (which will be satisfied in our numerics).

Assumption 3.2. *For all* $0 \le k \le T$, *we have* $Q(k,1) = 0$.

Then, starting from $I_0 = 1$, in the global model, it would be optimal for the bank to call the asset immediately, see (2.3) with $t = 0$. But the use of the local model may lead the trader to overvalue the claim and to a delayed exercise decision.

Remark 3.4. Playing with different numerical parametrizations of the model often leads to $Q(\cdot,1) \equiv 0$. In particular, for any positive parameter γ_{T-1}, forcing $Q(\cdot,1) = 0$ and the continuation value $-e^{-2\gamma_k} + u_{k+1}Q(k+1,1) + v_{k+1}Q(k+1,-1)$ to be 0 in the equation for $Q(k,1)$ in (3.9) yields $Q(T,\cdot) = 0$ and for decreasing $k \le T-1$,

$$\begin{aligned} Q(k,-1) &= e^{-2\gamma_k} + \frac{1}{2}(1+e^{-2\gamma_k})Q(k+1,-1),\\ 1 &= \frac{1}{2}\left(e^{2\gamma_{k-1}} - 1\right)Q(k,-1) \text{ i.e. } \gamma_{k-1} = \frac{1}{2}\ln\left(1 + \frac{2}{Q(k,-1)}\right),\end{aligned}$$

which iteratively determine $Q(k,-1) > 0$ and $\gamma_k > 0$. This provides a whole family of models for which $Q(\cdot,1) \equiv 0$ (i.e. Assumption 3.2 holds), parameterized by $\gamma_{T-1} > 0$.

Under Assumption 3.2, the only events that are relevant to the not-so-bad trader are the following partition of Ω:

$$\begin{aligned}&\Omega_{T+1,T+1} = \{I_0 = 1, \dots, I_T = 1\},\\ &\Omega_{l,T+1} = \{I_0 = 1, \dots, I_{l-1} = 1, I_l = -1, \dots, I_T = -1\}, \quad 1 \le l \le T,\\ &\Omega_{l,m} = \{I_0 = 1, \dots, I_{l-1} = 1, I_l = -1, \dots, I_{m-1} = -1, I_m = 1\}, \quad 1 \le l < m \le T,\end{aligned} \tag{3.26}$$

where $\Omega_{T+1,T+1}$ corresponds to the extreme event never occurring on $0 \dots T$; for $1 \le l \le T$, $\Omega_{l,T+1}$ corresponds to the extreme event first happening at time l and never ceasing on $l \dots T$; for $1 \le l < m \le T$, $\Omega_{l,m}$ corresponds to the extreme event first occurring at time l and then first ceasing at time m. Note that $\Omega_{l,m}$ is $\mathfrak{F}_m$ measurable, for each $1 \le l < m \le T$, and $\Omega_{l,T+1}$ is $\mathfrak{F}_T$ measurable, for $1 \le l \le T+1$.

The index-set of these market events is

$$\mathcal{I} := \{(l,m);\, 1 \le l < m \le T\} \cup \{(l,T+1);\, 1 \le l \le T+1\}.$$

For $(l,m) \in \mathcal{I}$, we have $\Omega_{l,m} \subset \Omega_l$ (compare (3.26) and (3.18)), $I_k(\Omega_{l,m}) = \mathbf{1}_{k<l} - \mathbf{1}_{l \le k < m} + \mathbf{1}_{k=m}$ for $k \le m \wedge T$, $\tau_s(\Omega_{l,m}) = l \wedge T$, and

$$\tau_e^{\mathrm{nsb}}(\Omega_{l,m}) = \tau_e^{\mathrm{bad}}(\Omega_{l,m}) \mathbb{1}_{\{\tau_e^{\mathrm{bad}}(\Omega_{l,m}) < l \wedge T\}} + \inf\{k \ge \tau_s(\Omega_{l,m});\, Q_k = 0\} \mathbb{1}_{\{\tau_e^{\mathrm{bad}}(\Omega_{l,m}) = l \wedge T\}}.$$

Also note that $\tau_e^{\mathrm{nsb}}(\Omega_{l,m}) \le m$ if $m \le T$ as, on $\{\tau_e^{\mathrm{bad}}(\Omega_{l,m}) = l\}$, $Q_m = Q(m, I_m(\Omega_{l,m})) = Q(m,1) = 0$ by Assumption 3.2. Moreover, with the notations (3.6) at hand, as proved in Sec. A.2.

Lemma 3.6. *For every $0 \le k \le T$, the $\mathfrak{F}_k$ conditional probabilities of the partitioning events $\Omega_{\lambda,\mu}$, $(\lambda,\mu) \in \mathcal{I}$, are constant on each $\Omega_{l,m}$, $(l,m) \in \mathcal{I}$, where they are worth*

$$\begin{aligned}
&\mathbb{Q}_k\left[\Omega_{\lambda,\mu}\right](\Omega_{l,m}) \\
&\quad = \left(\mathbf{1}_{k<l\wedge\lambda} + \mathbf{1}_{k\ge l\wedge\lambda}\mathbf{1}_{l=\lambda}(\mathbf{1}_{k<m\wedge\mu} + \mathbf{1}_{k\ge m\wedge\mu}\mathbf{1}_{m=\mu})\right) \\
&\quad\quad \times \left(\mathbf{1}_{k\ge\mu} + \mathbf{1}_{\lambda\le k<\mu}\left(\prod_{r=k+1}^{\mu-1} u_r\right) v_\mu + \mathbf{1}_{k<\lambda}\left(\prod_{r=k+1}^{\lambda-1} u_r\right) v_\lambda \left(\prod_{r=\lambda+1}^{\mu-1} u_r\right) v_\mu\right), \\
&\quad\quad 1 \le \lambda < \mu \le T, \\
&\mathbb{Q}_k\left[\Omega_{\lambda,T+1}\right](\Omega_{l,m}) \\
&\quad = \left(\mathbf{1}_{k<l\wedge\lambda} + \mathbf{1}_{k\ge l\wedge\lambda}\mathbf{1}_{l=\lambda}\mathbf{1}_{k<m}\right) \\
&\quad\quad \times \left(\mathbf{1}_{k\ge\lambda}\prod_{r=k+1}^{T} u_r + \mathbf{1}_{k<\lambda}\left(\prod_{r=k+1}^{\lambda-1} u_r\right) v_\lambda \left(\prod_{r=\lambda+1}^{T} u_r\right)\right), \\
&\quad\quad 1 \le \lambda \le T \quad \textit{and} \\
&\mathbb{Q}_k\left[\Omega_{T+1,T+1}\right](\Omega_{l,m}) = \mathbf{1}_{k<l}\prod_{r=k+1}^{T} u_r. \qquad (3.27)
\end{aligned}$$

Since the market is represented by the process I and the processes related to the not-so-bad trader are all stopped at τ_e^{nsb}, the study regarding the latter boils down to understanding computations relative to $\mathfrak{F}_{\tau_e^{\mathrm{nsb}}} \cap \sigma(I_k, k \le T)$ measurable random variables. Now, as proved in Sec. A.2.

Lemma 3.7. (i) *Let* ξ *be an* $\mathfrak{F}_{\tau_e^{\rm nsb}} \cap \sigma(I_k, K \leq T)$ *measurable random variable. Then* ξ *is constant on each* $\Omega_{l,m}$, $(l,m) \in \mathcal{I}$.

(ii) *Let* ξ *be a random variable constant on each* $\Omega_{l,m}$, $(l,m) \in \mathcal{I}$. *Then, for each* $0 \leq k \leq T$ *and* $(l,m) \in \mathcal{I}$, $\mathbb{E}_k[\xi]$, $\mathbb{V}\mathrm{a}\mathbb{R}_k(\xi)$ *and* $\mathbb{ES}_k(\xi)$ *are constant on each* $\Omega_{l,m}$; *in particular,*

$$\mathbb{E}_k[\xi](\Omega_{l,m}) = \sum_{(\lambda,\mu)\in\mathcal{I}} \xi(\Omega_{\lambda,\mu})\mathbb{Q}_k[\Omega_{\lambda,\mu}](\Omega_{l,m}). \tag{3.28}$$

We now apply these abstract results to the random variables associated with the not-so-bad trader. The proof is similar to the proof of Proposition 3.3 and thus omitted.

Proposition 3.4. *Let* $k \leq T$.

(i) *We have*

$$\begin{aligned} pnl_k^{\rm nsb} &= \mathcal{Q}_{k\wedge\tau_e^{\rm nsb}} + q_{k\wedge\tau_e^{\rm nsb}} \mathbb{1}_{\{k\wedge\tau_e^{\rm nsb}<\tau_s\}} + Q_{k\wedge\tau_e^{\rm nsb}} \mathbb{1}_{\{k\wedge\tau_e^{\rm nsb}\geq\tau_s\}} \\ &\quad - \left(\mathcal{P}^{\rm loc}_{k\wedge\tau_e^{\rm nsb}\wedge\tau_s} + P^{\rm nsb}_{k\wedge\tau_e^{\rm nsb}\wedge\tau_s}\right) \\ &\quad - \left(\mathcal{P}^{\rm fair}_{k\wedge\tau_e^{\rm nsb}} - \mathcal{P}^{\rm fair}_{\tau_s} + P^{\rm fair}_{k\wedge\tau_e^{\rm nsb}} - P^{\rm fair}_{\tau_s}\right)\mathbb{1}_{\{k\wedge\tau_e^{\rm nsb}\geq\tau_s\}}, \\ \mathrm{HVA}_k^{\rm nsb} &= \underbrace{\mathbb{1}_{\{k\wedge\tau_e^{\rm nsb}<\tau_s\}}(q_{k\wedge\tau_e^{\rm nsb}} - Q_{k\wedge\tau_e^{\rm nsb}})}_{=:U_k^{\rm nsb}} \\ &\quad + \underbrace{\mathcal{Q}_{k\wedge\tau_e^{\rm nsb}} + Q_{k\wedge\tau_e^{\rm nsb}} - \mathbb{E}_k\left[\mathcal{Q}_{\tau_e^{\rm nsb}} + Q_{\tau_e^{\rm nsb}}\right]}_{=va(K^{\tau_e^{\rm nsb}})_k}. \end{aligned} \tag{3.29}$$

(ii) *The random variables* $pnl_k^{\rm nsb}$ *and* $\mathrm{HVA}_k^{\rm nsb}$ *are constant on each of the* $\Omega_{l,m}$, $(l,m) \in \mathcal{I}$, *where their values can be computed by application of Propositions* 3.1 *and* 3.2 *and Lemmas* 3.6 *and* 3.7.

(iii) $\mathrm{EC}_k^{\rm nsb}$, *as defined in* (2.10) *specified to the not-so-bad trader dealing the range accrual, is constant on each of the* $\Omega_{l,m}$, $(l,m) \in \mathcal{I}$.

(iv) $\mathrm{KVA}_k^{\rm nsb}$, *as defined in* (2.10) *specified to the not-so-bad trader dealing the range accrual, is constant on each of the* $\Omega_{l,m}$, $(l,m) \in \mathcal{I}$. *In addition,*

$$\mathrm{KVA}_0^{\rm nsb} = h\sum_{l=0}^{T-1} e^{-hl} \sum_{(\lambda,\mu)\in\mathcal{I}} \max(\mathrm{KVA}_l^{\rm nsb}(\Omega_{\lambda,\mu}), \mathrm{EC}_l^{\rm nsb}(\Omega_{\lambda,\mu}))\mathbb{Q}[\Omega_{\lambda,\mu}].$$

Remark 3.5. In the $\mathrm{HVA}^{\rm nsb}$ equation in (3.29), we see no $V_k^{\rm nsb} := \mathbb{E}_k\left[Q_{\tau_e^{\rm nsb}}\mathbb{1}_{\{\tau_e^{\rm nsb}<\tau_s\}}\right]$ analog of the $V_k^{\rm bad}$ term in the $\mathrm{HVA}^{\rm bad}$ equation (3.23). This is because, on $\{\tau_e^{\rm nsb} < \tau_s\}$, $Q_{\tau_e^{\rm nsb}}$ vanishes by Assumption 3.2, hence $V_k^{\rm nsb} = 0$. We see no $W_k^{\rm nsb} := \mathbb{E}_k\left[\mathbb{1}_{\{\tau_e^{\rm nsb}\geq\tau_s\}}\mathcal{Q}_{\tau_e^{\rm nsb}}\right]$ analog of the $W_k^{\rm bad}$ term either because, on $\{\tau_e^{\rm nsb} \geq \tau_s\}$, $\mathcal{Q}_{\tau_e^{\rm nsb}} = 0$ holds by Definition 2.5, hence $W_k^{\rm nsb} = 0$.

Corollary 3.1 allows one to recalibrate the local model analytically conditionally on any scenario of the fair valuation model. Proposition 3.2 allows one to price analytically in the local model shifting along the fair valuation one. Propositions 3.3 and 3.4 allow one to compute the HVA and KVA of the bad and of the not-so-bad traders analytically in any scenario of the fair valuation model. All in one, the cost of computing the HVA and the KVA in this setup is reduced to the one of running the exact dynamic programming equations (3.9) for Q and (3.13) for each q^k, $0 \leq k \leq T$, along with companion analytical valuations at each nodes of the corresponding computational trees, of sizes $\mathrm{O}(T)$ each, hence a total computational cost in $\mathrm{O}(T^2)$, and exact computations (in our fully discrete setup we avoid the numerical error inherent to any PDE numerical or Monte Carlo simulation scheme). This simple but representative example illustrates all the ins and outs of recalibration risk and Darwinian model risk, while allowing us to understand how, conversely, such calculations would be unfeasible for a banking portfolio and realistic models: other types of callable assets could in principle be considered following the same logic, with expected similar qualitative insights, but a more complex setup would lead to much more involved computations, with nested numerical optimization for the embedded recalibration task in particular. Not only would this result into an extremely heavy procedure, but it would induce a numerical error obscuring the financial interpretation.

Remark 3.6. The practical relevance of the callable range accrual — a widely traded asset known to have caused significant losses — makes it an enlightening example. The probabilistic model that we consider, considering only a finite number of market scenarios, is tailored to the product. It allows one to recalibrate the local model analytically (without any numerical optimization) conditionally on any scenario of the reference model, to compute exactly (i.e. without numerical approximations with e.g. PDE methods or Monte-Carlo simulations) all the quantities of interest, and to provide financial interpretations and recommendations.

4. Numerical Results

We take $T = 10$ years and $\gamma_k = \int_k^{k+1} \gamma(s)ds$ with

$$\gamma(s) := 0.15 - \frac{0.1s}{T} = 0.15 - 0.01s. \tag{4.1}$$

Hence $\gamma_k = 0.15 - \frac{0.1}{2T}((k+1)^2 - k^2) = 0.15 - \frac{0.1}{2T}(2k+1)$, for $0 \leq k \leq T-1$. The jump intensity functions $(\gamma_l)_{l \leq T-1}$ and $(\nu_l^0)_{l \leq T-1}$ calibrated to it via (3.4) for $k = 0$ are represented in Fig. 1.

A nominal (scaling factor) of 100 is applied everywhere to ease the readability of the results. Figure 2 displays the pricing functions $Q(t, \mp 1)$ and $q^0(t, \mp 1)$ of the callable range accrual in the fair valuation model and in the trader's local model calibrated to it at time 0, computed by the dynamic programming equations of Propositions 3.1 and 3.2. The trader's local model overvalues the option, which

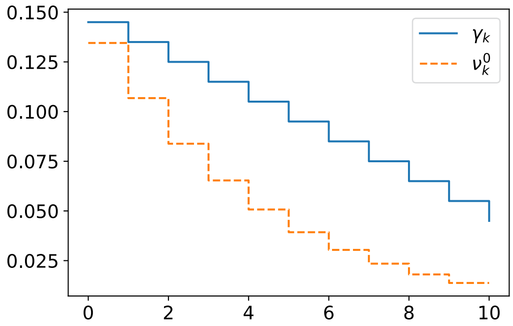

Fig. 1. Fair model jump intensity function $(\gamma_k)_{k \le T-1}$ and local model jump intensity function $(\nu_k^0)_{k \le T-1}$ calibrated on binaries at $t = 0$.

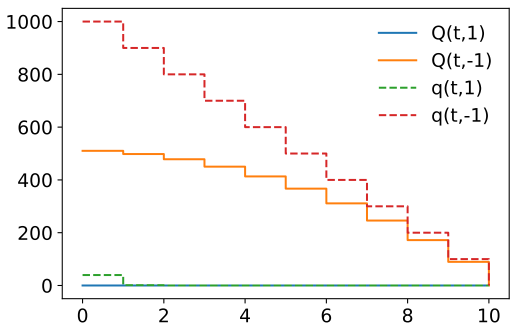

Fig. 2. Pricing functions of the range accrual in the global model, $Q(\cdot,\cdot)$, and in the local model calibrated on binaries at time 0, $q^0(\cdot,\cdot)$.

increases his competitiveness for buying the claim from his client, in line with the first Darwinian principle of Sec. 1.

Note that the pricing function $Q(\cdot,1)$ satisfies Assumption 3.2. Hence, based on Propositions 3.3 and 3.4 and their consequences detailed in Secs. 3.2 and 3.3, one has numerically access to an exhaustive description of both cases at hand (the bad trader as per Sec. 3.2 and the not-so-bad trader under Assumption 3.2 as per Sec. 3.3), exact within machine precision (only involving discrete dynamic programming equations or exact formulas for path-dependent quantities, without Monte Carlo simulations). Figure 3 shows the time-0 hedging ratios in the binary options struck along the lower barrier $i = -1$ and Fig. 4 shows the time-0 hedging ratios in the binary options struck along the upper barrier $i = +1$.

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4.1. *Bad trader*

For $\gamma(\cdot)$ as per (4.1), the dynamic programming equations yield $q^1(1,1) > 0$ and $q^2(2,1) = 0$. The first equality implies that the trader calls back the option at $t = 1$ if and only if the model switch occurs at $t = 1$. If $\tau_s > 1$, then the trader always

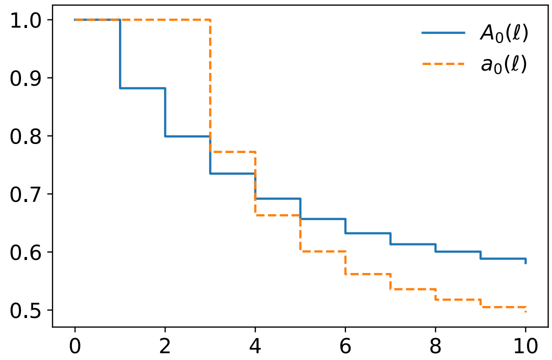

Fig. 3. Hedging ratios in the binaries struck along the lower barrier $i = -1$ in the global model, $A_0(\ell)$, and in the local model calibrated to the fair values of all binaries at time 0, $a_0(\ell)$.

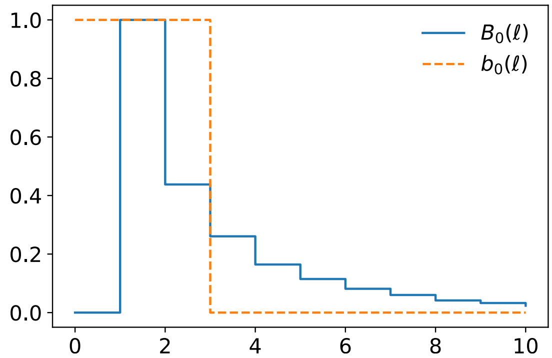

Fig. 4. Hedging ratios in the binaries struck along the upper barrier $i = +1$ in the global model, $B_0(\ell)$, and in the local model calibrated to the fair values of all binaries at time 0, $b_0(\ell)$.

calls the asset at $t = 2$, whether that $I_2 = -1$, i.e. $\tau_s = 2$, or that $I_2 = 1$ and as $q^2(2,1) = 0$, it is optimal for the bad trader to exercise. Hence the only relevant events are Ω_1, Ω_2 and $\Omega_{T+1=11}$ (on each Ω_l, $l \geq 3$, everything happens as on Ω_{11}).

Figure 5, *center panel*, displays pnl^{bad} on these events. We decompose pnl^{bad} in two terms corresponding to the two lines for pnl_k^{bad} in (3.23): the cash flows of the first line resulting from holding the option and its hedge plus the corresponding prices (*top panel*) and the ones of the second line accounting for calling the option at zero recovery (*bottom panel*). In the scenarios Ω_1 and Ω_2, where the asset is called due to the model switch, a profit (Fig. 5, *top panel*, analyzed in more detail in Sec. 4.1.1 below), is more than compensated by calling the asset, highly valuable at that moment (Fig. 5, *bottom panel*), resulting in an overall loss at the model switch time (Fig. 5, *center panel*).

Figure 6 displays the process HVA^{bad} (*top left panel*) and its split into three contributions (cf. the decomposition of $\text{HVA}_k^{\text{bad}}$ in (3.23), see also Remark 2.2): the

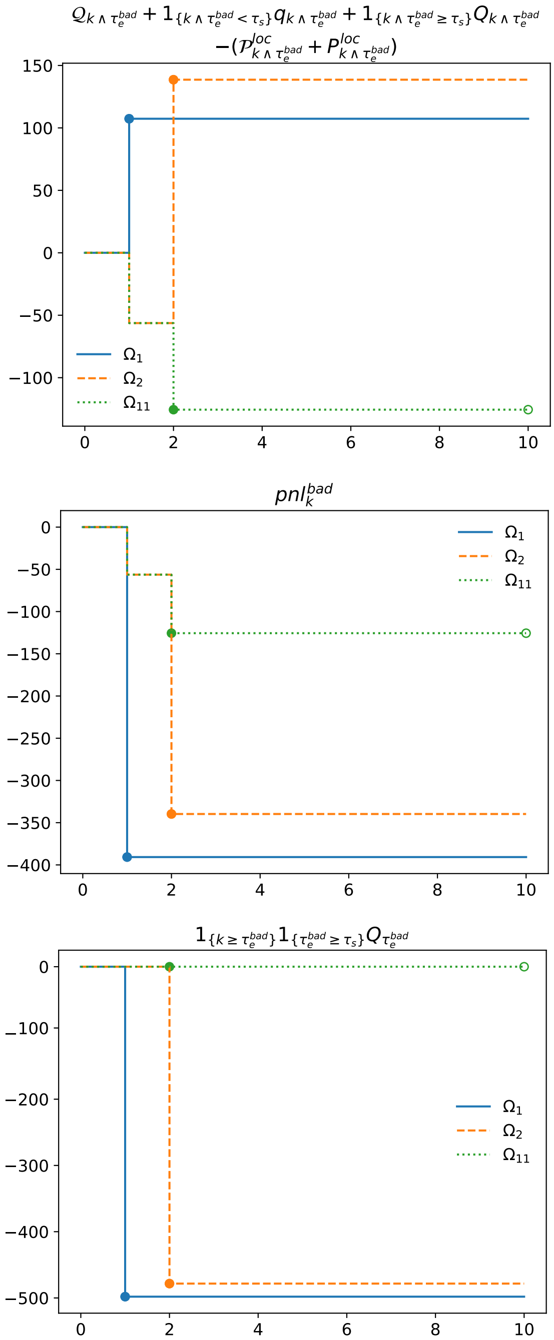

Fig. 5. (*Center*) bad trader's *pnl*, (*top*) callable option cash flow and price minus its hedge cash flow and price, (*bottom*) term accounting for calling the product (at zero recovery).

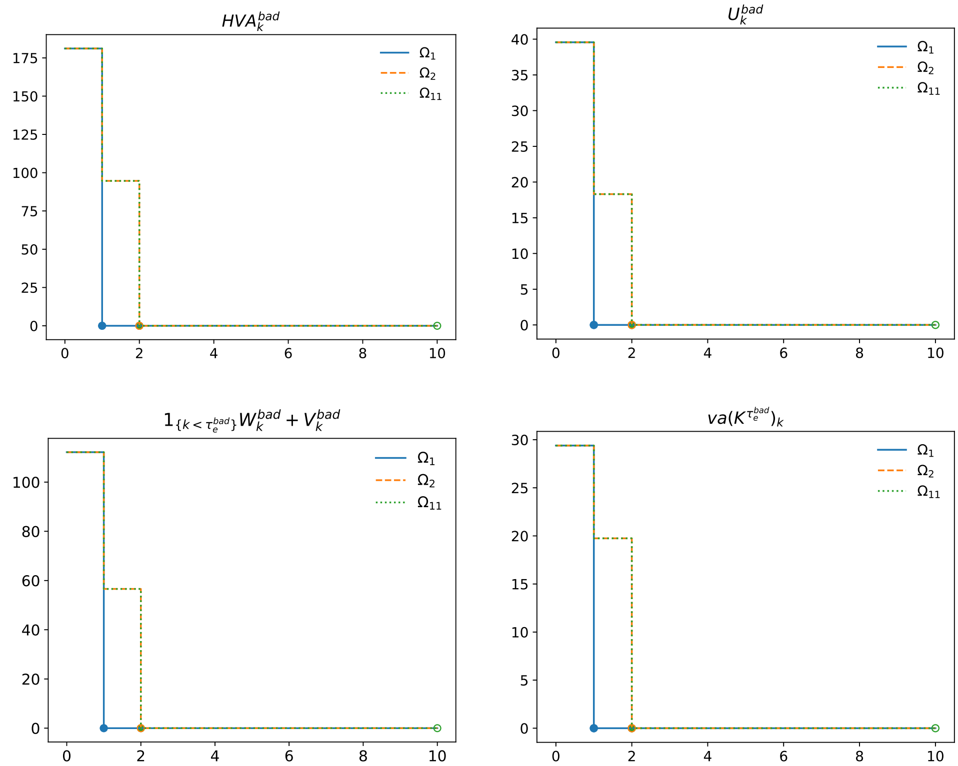

Fig. 6. Bad trader's HVA and its components.

misvaluation term U_k^{bad} when the trader uses his local model instead of the global one (*top right*), the expected cost of calling the asset at zero recovery $V_k^{\mathrm{bad}} + \mathbb{1}_{\{k<\tau_e^{\mathrm{bad}}\}} W_k^{\mathrm{bad}}$ (*bottom left*), and the reserve for suboptimal exercise $va(K^{\tau_e^{\mathrm{bad}}})_k$ (*bottom right*). By comparing the top left and right panels, we observe that the HVA on a callable claim can thus be several times greater than the price difference $q - Q$.

Figure 7 displays the HVA compensated pnl process of the bad trader. We notice that on the event Ω_{11}, where there is no switch and the trader calls back the claim at time 2, the gains resulting from the depreciation of the HVA cover the *pnl* losses (the green curve is in the negative), in line with the second Darwinian principle recalled in Sec. 1. But, on Ω_1 and Ω_2, the losses made at τ_s supersede the systematic profits made before τ_s, in line with the third Darwinian principle of Sec. 1.

4.1.1. *Detailed understanding of the profit in Fig.* 5, *top panel*

On $\Omega_1 \cup \Omega_2 = \{\tau_s = \tau_e^{\mathrm{bad}}\}$, at τ_s, the bank gets on the asset a cash flow $\mathcal{Q}_{\tau_s} - \mathcal{Q}_{\tau_s-} = 1$, while it pays on the static hedge a cash flow $\mathcal{P}^{\mathrm{loc}}_{\tau_s} - \mathcal{P}^{\mathrm{loc}}_{\tau_s-} = a_0(\tau_s)$, as $I_{\tau_s} = -1$. In addition, in any time-τ_s (hence, no longer calibrated) local model and independently

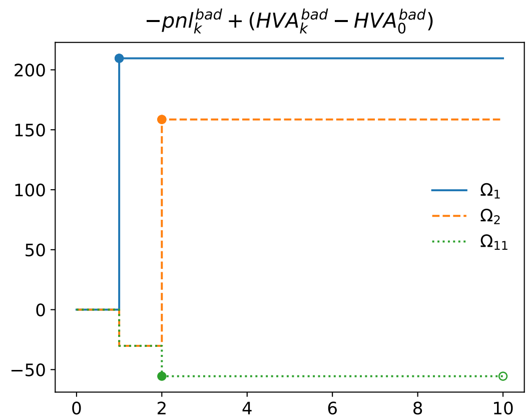

Fig. 7. HVA-compensated loss-and-profits of the bad trader.

Table 1. The decomposition (4.2) on the events Ω_1 and Ω_2.

	Ω_1	Ω_2
$1 + (T - \tau_s) - q^{\tau_s - 1}(\tau_s - 1, 1) - \left(a_0(\tau_s) + \sum_{k=\tau_s+1}^{T} a_0(k) - P^{\mathrm{loc}}_{\tau_s - 1}\right)$	335	391
$Q(\tau_s, -1) - (T - \tau_s) - \left(P^{\mathrm{loc}}_{\tau_s} - \sum_{k=\tau_s+1}^{T} a_0(k)\right)$	−227	−196

of the intensity function $\nu^{\tau_s}(\cdot)$, as $i^{\tau_s}_{\tau_s} = I_{\tau_s} = -1$ is an absorbing state, at time τ_s, the asset is worth $q_{\tau_s} = q^{\tau_s}_{\tau_s} = T - \tau_s$ and the hedge is worth $P^{\mathrm{loc}}_{\tau_s} = \sum_{k=\tau_s+1}^{T} a_0(k)$ (cf. Eqs. (2.5) and (3.16)). Hence the profit at the model switch time $\tau_s = 1$ or 2 made before calling the asset, as observed on the top panel of Fig. 5, can be decomposed as follows (see Table 1):

$$\begin{aligned}
&\left(\mathcal{Q}_{\tau_s} + Q_{\tau_s} - (\mathcal{P}^{\mathrm{loc}}_{\tau_s} + P^{\mathrm{loc}}_{\tau_s})\right) - \left(\mathcal{Q}_{\tau_s - 1} + q_{\tau_s - 1} - (\mathcal{P}^{\mathrm{loc}}_{\tau_s - 1} + P^{\mathrm{loc}}_{\tau_s - 1})\right) \\
&\quad = 1 + (T - \tau_s) - q^{\tau_s - 1}(\tau_s - 1, 1) - \left(a_0(\tau_s) + \sum_{k=\tau_s+1}^{T} a_0(k) - P_{\tau_s - 1}\right) \\
&\qquad + Q(\tau_s, -1) - (T - \tau_s) - \left(P^{\mathrm{loc}}_{\tau_s} - \sum_{k=\tau_s+1}^{T} a_0(k)\right), \qquad (4.2)
\end{aligned}$$

where the third line corresponds to the change of valuation model at τ_s, which is a loss as per Fig. 2. An overall profit (made, at least, *before* calling the asset) means that this loss is more than compensated by a profit coming from the second line, coming from the static hedge not being perfect, especially at τ_s (from τ_s onward, the perfect hedge would be to short a digital option with payoff $\mathbb{1}_{\{I_k = -1\}}$ for each $k > \tau_s$).

Table 2. Time-0 HVA and KVA for parallel shocks s on the jump intensity in the fair valuation model, together with finite-differences approximations of the corresponding sensitivities around the baseline scenario $s = 0$.

Shock s	HVA_0^s	KVA_0^s	$\dfrac{\mathrm{HVA}_0^s - \mathrm{HVA}_0^0}{s}$	$\dfrac{\mathrm{KVA}_0^s - \mathrm{KVA}_0^0}{s}$
0	181.125	35.891		
0.00050	181.600	35.771	950.849	−239.752
0.00025	181.363	35.831	951.344	−240.055
−0.00050	180.648	36.011	952.829	−240.969
−0.00025	180.887	35.951	952.334	−240.663

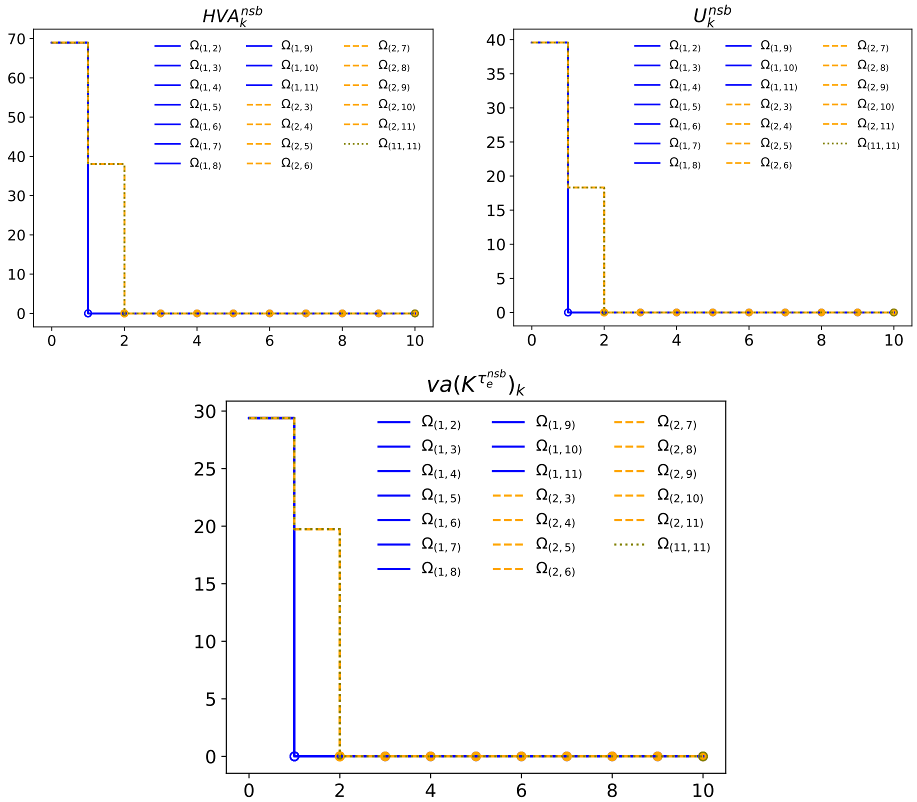

Fig. 8. Not-so-bad trader's HVA and its components.

4.1.2. *Numerical stability of the valuation adjustments*

To assess the numerical stability of the proposed metrics, we study the impact on the valuation adjustments HVA_0 and KVA_0 of a parallel shift of the jump intensity $(\gamma_k)_{k \le T-1}$ (cf. (4.1)) in the fair valuation model. This also allows us to compute finite-difference approximations of the sensitivities of HVA_0 and KVA_0 with respect to such parallel shifts.

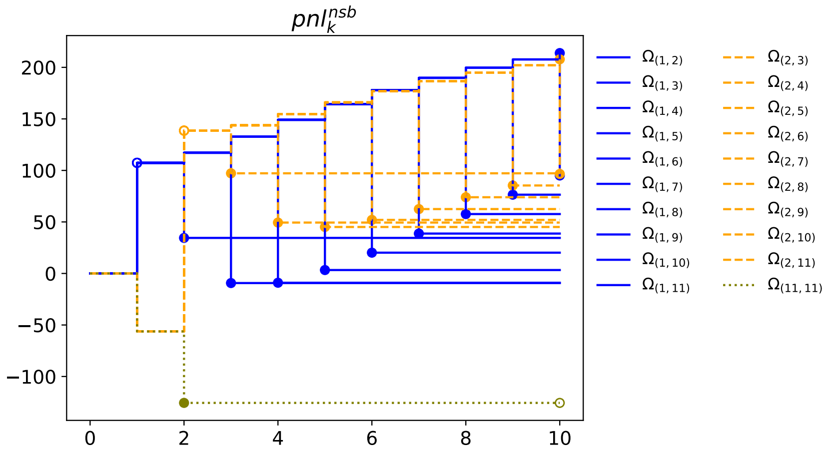

Fig. 9. Not-so-bad trader's *pnl*.

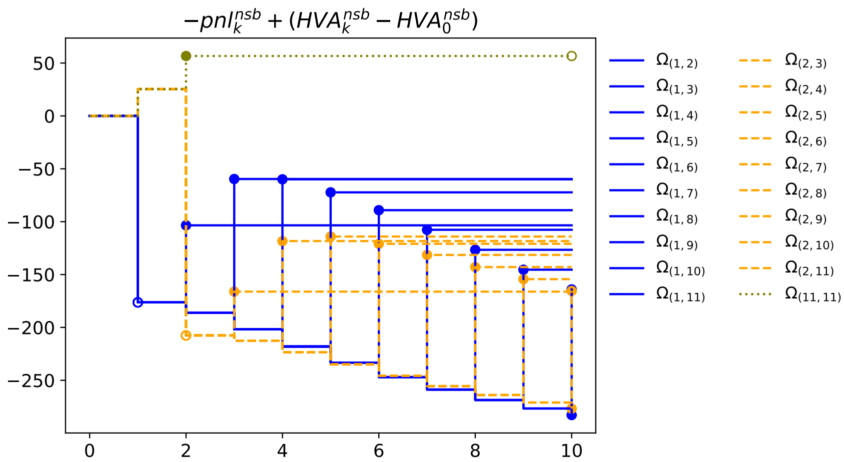

Fig. 10. HVA-compensated loss-and-profits of the not-so-bad trader.

Namely, given a small (real) shift s, we consider the shocked jump intensity $\gamma_k^s := \gamma_k + s$, $0 \leq k \leq T-1$ and we compute the corresponding valuation adjustments HVA_0^s and KVA_0^s using the numerical procedure described above in the baseline case $s = 0$. We then deduce the sensitivities around the baseline case by finite-difference approximations. The results are reported in Table 2. We observe that the valuation adjustments remain stable around the baseline scenario, and that the corresponding finite-difference estimates of the sensitivities are themselves stable.

These sensitivities are directional derivatives in a prescribed direction. A more holistic notion of sensitivity would be obtained by considering an upsilon (Υ) sensitivity a la Bartl *et al.* (2021), which we leave for future research.

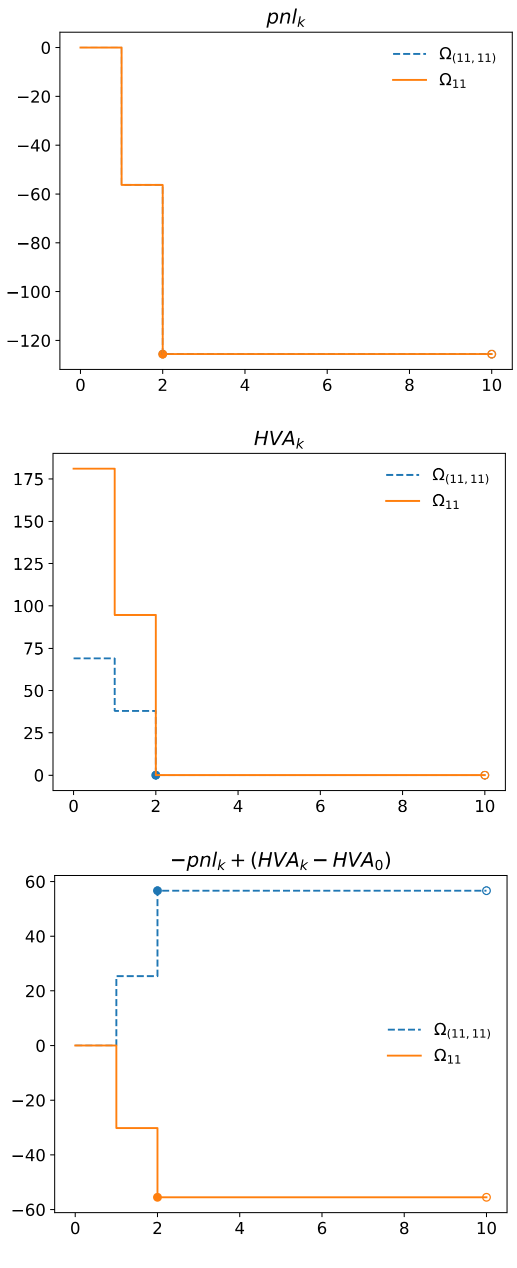

Fig. 11. (*top*) *pnl*, (*center*) HVA, and (*bottom*) $\mathrm{HVA}_{(0)} - pnl$ of the bad trader and the not-so-bad trader in the absence of model switch.

4.2. *Not-so-bad trader*

Regarding the not-so-bad trader, as $q^2(2,1) = 0$, the option is called at $k = 2$ if the model switch has not occurred before, hence all the $\Omega_{l,m},\ 3 \leq l \leq 10$, are equivalent to $\Omega_{11,11}$. As for $l \leq 2$, on $\Omega_{l,m}$, the not-so-bad trader always calls the option at time m, which is the first time beyond l for which $Q(m, I_m) = Q(m, 1) = 0$. Accordingly, we only report on the results corresponding to the events $\Omega_{l,m}$, for $l = 1$ or 2 and $m > l$, and $\Omega_{11,11}$.

Figure 8 displays the not-so-bad trader's HVA (*top left*) and its split in valuation (bottom) and early callability (*top right*) components (see Proposition 3.4(i)). Comparing with the bad trader's HVA components displayed in Fig. 6, we only see here U^{nsb} and $va(K^{\tau_e^{\mathrm{nsb}}})$ components, as the analogous processes V^{nsb} and W^{nsb} vanish as already observed in Remark 3.5.

The comparison with the top left panel of Fig. 6 shows that $\mathrm{HVA}^{\mathrm{nsb}}$ is more than twice smaller than $\mathrm{HVA}^{\mathrm{bad}}$, but still significantly greater than the price difference $(q - Q)\mathbb{1}_{[0,\tau_s)}$ (see the top left panel of Fig. 8).

Figures 9 and 10 display the not-so-bad trader's *pnl* and HVA compensated *pnl* process. As opposed to what we saw in Fig. 7 regarding the bad trader, on the event $\Omega_{11,11}$, where there is no model switch and the not-so-bad trader calls back the claim according to the prescriptions of his wrong model, the gains resulting from the depreciation of the HVA no longer cover the *pnl* losses (the dotted curve is in the positive in Fig. 10): the better practice of switching to the global model once the trader's local model no longer calibrates not only diminishes the HVA, but also avoids the short-to-medium term incentives to use the local model. In fact, the local model does not pass the second Darwinian principle for the not-so-bad trader (see Sec. 1), and would therefore not be selected by the latter (but only by the bad trader).

Figure 11 gathers on the same page the previous results for both traders in the event where the switch never happens, i.e. on Ω_{11} in the case of the bad trader and on $\Omega_{11,11}$ in the case of the not-so-bad one. The corresponding paths of the *pnl* appear to be identical (as they indeed are) in the top panel of Fig. 11. As explained above, the HVA of the not-so-bad trader is smaller than the one of the bad one (*middle panel*); the HVA depreciation gains of the bad trader fakely more than compensate his raw *pnl* losses (fakely in the sense that these systematic gains in fact only compensate future losses), but this is not the case for the not-so-bad trader (top and bottom panels).

5. Conclusion

Figures 12 and 13 show the economic capital processes of the bad and not-so-bad traders of Propositions 3.3–3.4(iii), resulting in the KVA_0 displayed for a hurdle rate h of 10% in Table 3, along with the corresponding HVA_0. As expected, $\mathrm{HVA}_0^{\mathrm{nsb}} \leq \mathrm{HVA}_0^{\mathrm{bad}}$ and $\mathrm{KVA}_0^{\mathrm{nsb}} \leq \mathrm{KVA}_0^{\mathrm{bad}}$, which illustrates the relevance of the proposed HVA and KVA metrics in terms of their sensitivities to the specification of the setup. In this example, the KVA is largely dominated by the HVA, by a factor > 4, whereas the opposite was prevailing in the case of model risk on a European claim in Bénézet & Crépey (2024, Eq. (37)). However, a common and salient conclusion

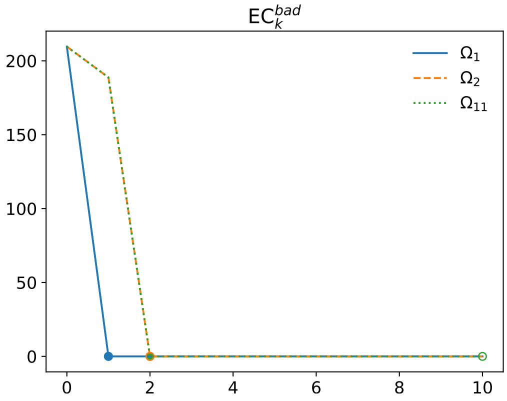

Fig. 12. Bad trader's economic capital.

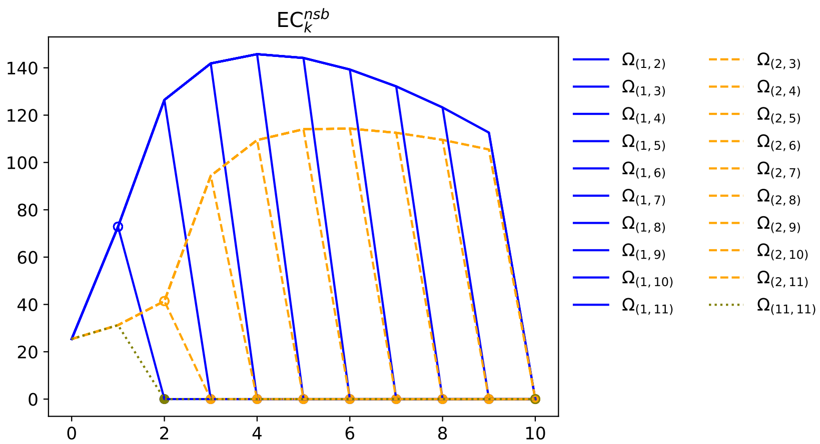

Fig. 13. Not-so-bad trader's economic capital.

Table 3. HVA_0 and KVA_0 of the traders.

	HVA_0	KVA_0
Bad trader	181	36
Not-so-bad trader	69	15

is that, in all the considered examples: bad or not-so-bad trader dealing a callable claim here or a European claim (for which bad or not-so-bad was in fact the same) in the previous paper, the risk-adjusted HVA, AVA = HVA + KVA (additional valuation adjustment for model risk), is much larger than the price difference $q - Q$ of the claim between the trader's model and a reference model. Whether this is mainly due to an HVA effect as in the present callable case (see Figs. 6 and 8) or to a KVA effect in Bénézet & Crépey (2024), in any case, it provides quantitative arguments in favor of a reserve for model risk that should be much larger than the common practice of reserving such a price difference simply (cf. Bénézet & Crépey (2024, Remark 2.10)).

This paper is focused on the Darwinian model risk of adverse selection by traders of local models motivated by short-to-medium gains at the expense of long term losses. We demonstrate how this can be a critical model risk issue regarding the handling of structured products by banks. This holds even disregarding the uncertainties, most commonly considered in the academic model risk literature and simply ignored for clarity in this work, regarding the risk-neutral and physical probability measures that underlie our fininsurance (global valuation) measure $\mathbb{Q}$ (see Sec. 1.1). We refer to Bartl *et al.* (2021), specifically their Upsilon (Υ) sensitivity, see also Sauldubois & Touzi (2024), to assess quantitatively such uncertainties. In particular, our framework assumes access to a well-specified fair valuation model on which local models can be perfectly calibrated (or not anymore, at time τ_s). Some insights into the sensitivity of our HVA and KVA metrics are provided in the paper by the consideration of the two traders, the bad and not-so-bad one, and the assessment of the impact of their different behavior on the HVA and the KVA, as well as by the numerical stability study of Sec. 4.1.2. A more systematic investigation of HVA and KVA sensitivities, as well as the incorporation of the uncertainty on the underlying physical and risk-neutral measures, are left for future research.

An important overarching question is: How far do we go in adding valuation adjustments? A distinguishing Darwinian model risk feature is that it cannot be detected by standard market risk metrics such as value-at-risk, expected shortfall or stressed value-at-risk. Indeed Darwinian model risk derives from the cumulative effect of daily recalibrations and feeds into the first moment of returns (alpha leakages); the usual market risk metrics, instead, all focus on higher moments of return distributions at short-time horizons (such as one day). Darwinian model risk can only be seen by simulating the hedging behavior of a bad model within the environment of good model. Even under the elementary probabilistic model under consideration in Secs. 3 and 4, the computation of the valuation adjustments is nontrivial due to the nested recalibration of the local model at each node of the fair valuation one. So not only risk-adjusted reserve for model risk can be very high, but doing this for real portfolios and models, for which the (re)calibration can only be done numerically (as opposed to formulaically and exactly in our setup), would be far too demanding. We then do not advocate the banks to implement the HVA as an actual reserve on top of the regulatory requirements already implemented and which would affect their capital allocation strategies and lead to numerical and regulatory challenges. The primary aim of our work is not to propose immediate practical implementation, but rather to highlight an important risk: model risk inherent in using suboptimal or inadequate models is significantly greater than the pricing differences between good and bad models alone might suggest. Our methodology dissects this risk and makes explicit the hidden costs incurred. But we rather view the HVA as a theoretical market discipline and a warning signal, encouraging the adoption of better model standards before practical implementation challenges arise. The best practice would be that banks be more strongly incentivized by regulators to only rely on high-quality models, so that such overwhelming computations (beyond tailor-made examples such as the one of this paper) are simply not needed.

Appendix A. Proofs of the Combinatorial Lemmas of Sec. 3

A.1. *Bad trader*

Proof of Lemma 3.4. For $0 < \lambda \le k$, Ω_λ is $\mathfrak{F}_\lambda$ measurable, hence $\mathfrak{F}_k$ measurable, thus $\mathbb{Q}_k[\Omega_\lambda] = \mathbb{1}_{\Omega_\lambda}$; in addition, for each $0 \le l \le T+1$,

$$\mathbb{1}_{\Omega_\lambda}(\Omega_l) = \mathbf{1}_{l=\lambda}.$$

This proves

$$\mathbf{1}_{k\ge\lambda>0}\mathbb{Q}_k[\Omega_\lambda](\Omega_l) = \mathbf{1}_{k\ge\lambda>0}\mathbf{1}_{l=\lambda}.$$

Moreover, for each $0 \le k < \lambda \le T$, we compute

$$\begin{aligned}
\mathbb{Q}_k[\Omega_{T+1}] &= \prod_{m=1}^{k} \mathbb{1}_{\{N_m - N_{m-1}\ \text{even}\}} \prod_{m=k+1} \mathbb{Q}[N_m - N_{m-1}\ \text{even}] \\
&= \prod_{m=1}^{k} \mathbb{1}_{\{N_m - N_{m-1}\ \text{even}\}} \prod_{m=k+1} u_m, \\
\mathbb{Q}_k[\Omega_\lambda] &= \left(\prod_{m=1}^{k} \mathbb{1}_{\{N_m - N_{m-1}\ \text{even}\}}\right) \mathbb{Q}[N_\lambda - N_{\lambda-1}\ \text{odd}] \\
&\quad \times \prod_{m=k+1}^{\lambda-1} \mathbb{Q}[N_m - N_{m-1}\ \text{even}] \\
&= \prod_{m=1}^{k} \mathbb{1}_{\{N_m - N_{m-1}\ \text{even}\}} \left(\prod_{m=k+1}^{\lambda-1} u_m\right) v_\lambda,
\end{aligned}$$

where, for each $0 \le l \le T+1$, $\big(\prod_{m=1}^{k} \mathbb{1}_{\{N_m - N_{m-1}\ \text{even}\}}\big)(\Omega_l) = \mathbf{1}_{l>k}$. This proves $\mathbf{1}_{0\le k<\lambda}\mathbb{Q}_k[\Omega_\lambda](\Omega_l) = \mathbf{1}_{0\le k<\lambda}\mathbf{1}_{l>k}\big(\prod_{m=k+1}^{\lambda-1} u_m\big)v_\lambda$ as well as the last line in (3.20). □

Proof of Lemma 3.5. (i) From (3.19), τ_e^{bad} is a stopping time with respect to the filtration $\mathfrak{F}^I$. Moreover, ξ is measurable with respect to $\mathfrak{F}^I_{\tau_e^{\text{bad}}}$, hence $\xi\mathbb{1}_{\{\tau_e^{\text{bad}} \le l\}}$ is $\mathfrak{F}^I_l$ measurable, for each $1 \le l \le T$. Therefore, for each $1 \le l \le T$, $\xi\mathbb{1}_{\{\tau_e^{\text{bad}} \le l\}} = \Psi_l(I_0, \ldots, I_l)$ holds for some map $\Psi_l : \{1, -1\}^{l+1} \to \mathbb{R}$.

Note that $\tau_e^{\text{bad}} \le \tau_s = l$ holds on Ω_l, i.e. $\Omega_l \subseteq \{\tau_e^{\text{bad}} \le l\}$.

For $\omega \in \Omega_l$, we thus have $\xi(\omega) = \xi(\omega)\mathbb{1}_{\{\tau_e^{\text{bad}}(\omega) \le l\}} = \Psi_l(I_0(\omega), \ldots, I_l(\omega)) = \Psi_l(1, \ldots, 1, -1)$, hence $\xi(\Omega_l)$ is well defined for $1 \le l \le T$.

Similarly, for all $\omega \in \Omega_{T+1}$, one has $\xi(\omega) = \Psi_T(1, \ldots, 1)$, hence $\xi(\Omega_{T+1})$ is also well defined.

(ii) Since ξ is constant on each Ω_λ, $1 \leq \lambda \leq T + 1$, which partition Ω, the $\mathfrak{F}_k$ conditional law of ξ is given, for all $0 \leq k \leq T$, by

$$\mathcal{L}_k(\xi) = \sum_{\lambda=1}^{T+1} \mathbb{Q}_k[\Omega_\lambda]\boldsymbol{\delta}_{\xi(\Omega_\lambda)}. \tag{A.1}$$

By Lemma 3.4, $\mathbb{Q}_k[\Omega_\lambda]$ is constant on each Ω_l, implying that $\mathcal{L}_k(\xi)$ is also constant on each Ω_l, $1 \leq l \leq T$. In particular, $\mathbb{E}_k[\xi], \mathbb{V}a\mathbb{R}_k(\xi)$ and $\mathbb{ES}_k(\xi)$ are constant on each Ω_l. We also compute

$$\mathbb{E}_k[\xi] = \sum_{\lambda=1}^{T+1} \xi(\Omega_\lambda)\mathbb{Q}_k[\Omega_\lambda] = \sum_l \mathbb{1}_{\Omega_l} \sum_{\lambda=1}^{T+1} \xi(\Omega_\lambda)\mathbb{Q}_k[\Omega_\lambda](\Omega_l).$$

Last, if $l \leq k$, then Lemma 3.4 yields $\mathbb{Q}_k[\Omega_\lambda](\Omega_l) = \mathbf{1}_{\lambda=l}$, hence (A.1) reduces to

$$\mathcal{L}_k(\xi)(\Omega_l) = \mathbb{Q}_k[\Omega_l](\Omega_l)\boldsymbol{\delta}_{\xi(\Omega_l)} = \boldsymbol{\delta}_{\xi(\Omega_l)},$$

which implies that $\mathbb{E}_k[\xi](\Omega_l) = \mathbb{V}a\mathbb{R}_k[\xi](\Omega_l) = \mathbb{ES}_k[\xi](\Omega_l) = \xi(\Omega_l)$. □

A.2. *Not-so-bad trader*

Proof of Lemma 3.6. For each $(l, m) \in \mathcal{I}$, all paths of I represented in $\Omega_{l,m}$ have the same beginning until time step $m \wedge T$. We denote by $\Omega^k_{l,m}$ the event defined by this beginning of the path of I until time step $k \leq m \wedge T$.

We compute

$$\begin{aligned}\mathbb{Q}_k[\Omega_{T+1,T+1}] &= \prod_{r=1}^{T} \mathbb{Q}_k[N_r - N_{r-1} \text{ even}] \\ &= \prod_{r=1}^{k} \mathbb{1}_{\{N_r - N_{r-1} \text{ even}\}} \prod_{r=k+1}^{T} \mathbb{Q}[N_r - N_{r-1} \text{ even}] \\ &= \mathbb{1}_{\Omega^k_{T+1,T+1}} \prod_{r=k+1}^{T} u_r,\end{aligned}$$

where $\mathbb{1}_{\Omega^k_{T+1,T+1}} = \sum_{1 \leq l \leq m \leq T} \mathbb{1}_{\Omega_{l,m}} \mathbf{1}_{k \leq l} + \sum_{l \leq T} \mathbb{1}_{\Omega_{l,T+1}} \mathbf{1}_{k<l} + \mathbb{1}_{\Omega_{T+1,T+1}}$, which proves the last identity in (3.27).

Similarly, for $1 \leq \lambda \leq T$,

$$\mathbb{Q}_k[\Omega_{\lambda,T+1}] = \mathbb{1}_{\Omega^k_{\lambda,T+1}} \times \left(\mathbf{1}_{k \geq \lambda} \prod_{r=k+1}^{T} u_r + \mathbf{1}_{k<\lambda} \left(\prod_{r=k+1}^{\lambda-1} u_r\right) v_\lambda \left(\prod_{r=\lambda+1}^{T} u_r\right)\right),$$

where

$$\mathbb{1}_{\Omega^k_{\lambda,T+1}}(\Omega_{l,m}) = \mathbf{1}_{1\le l\le m\le T}(\mathbf{1}_{k<l\wedge\lambda} + \mathbf{1}_{k\ge l\wedge\lambda}\mathbf{1}_{l=\lambda}\mathbf{1}_{k<m}) + \mathbf{1}_{1\le l\le T,m=T+1}(\mathbf{1}_{k<l\wedge\lambda} + \mathbf{1}_{k\ge l\wedge\lambda}\mathbf{1}_{l=\lambda}) + \mathbf{1}_{l=T+1,m=T+1}\mathbf{1}_{k<l\wedge\lambda} = \mathbf{1}_{k<l\wedge\lambda} + \mathbf{1}_{k\ge l\wedge\lambda}\mathbf{1}_{l=\lambda}\mathbf{1}_{k<m},$$

which proves the second identity in (3.27).

Finally, for $1 \le \lambda \le \mu \le T$,

$$\mathbb{Q}_k\left[\Omega_{\lambda,\mu}\right](\Omega_{l,m}) = \mathbb{1}_{\Omega^k_{\lambda,\mu}}\left(\mathbf{1}_{k\ge\mu} + \mathbf{1}_{\lambda\le k<\mu}\left(\prod_{r=k+1}^{\mu-1} u_r\right)v_\mu + \mathbf{1}_{k<\lambda}\left(\prod_{r=k+1}^{\lambda-1} u_r\right)v_\lambda\left(\prod_{r=\lambda+1}^{\mu-1} u_r\right)v_\mu\right),$$

where

$$\mathbb{1}_{\Omega^k_{\lambda,\mu}}(\Omega_{l,m}) = \mathbf{1}_{1\le l\le m\le T}\left(\mathbf{1}_{k<l\wedge\lambda} + \mathbf{1}_{k\ge l\wedge\lambda}\mathbf{1}_{l=\lambda}(\mathbf{1}_{k<m\wedge\mu} + \mathbf{1}_{k\ge m\wedge\mu}\mathbf{1}_{m=\mu})\right) + \mathbf{1}_{1\le l\le T,m=T+1}(\mathbf{1}_{k<l\wedge\lambda} + \mathbf{1}_{k\ge l\wedge\lambda}\mathbf{1}_{l=\lambda}\mathbf{1}_{k<\mu}) + \mathbf{1}_{l=T+1,m=T+1}\mathbf{1}_{k<\lambda} = \mathbf{1}_{k<l\wedge\lambda} + \mathbf{1}_{k\ge l\wedge\lambda}\mathbf{1}_{l=\lambda}(\mathbf{1}_{k<m\wedge\mu} + \mathbf{1}_{k\ge m\wedge\mu}\mathbf{1}_{m=\mu}),$$

which proves the first identity in (3.27). □

Proof of Lemma 3.7. (i) Since τ_e^{nsb} is an $\mathfrak{F}^I$ stopping time and ξ is $\mathfrak{F}^I_{\tau_e^{\mathrm{nsb}}}$ measurable, it follows that $\xi\mathbb{1}_{\{\tau_e^{\mathrm{nsb}}\le m\}}$ is $\mathfrak{F}^I_m$ measurable, for each $1 \le m \le T$. We thus have, for all $1 \le m \le T$, $\xi\mathbb{1}_{\{\tau_e^{\mathrm{nsb}}\le m\}} = \Psi_m(I_0,\ldots,I_m)$ for some map $\Psi_m : \{1,-1\}^{m+1} \to \mathbb{R}$.

For $\omega \in \Omega_{l,m}$ such that $1 \le l < m \le T$, we have $\tau_e(\omega) \le m$, i.e. $\Omega_{l,m} \subseteq \{\tau_e^{\mathrm{nsb}} \le m\}$, and hence $\xi(\omega) = \xi(\omega)\mathbb{1}_{\{\tau_e^{\mathrm{nsb}}(\omega)\le m\}} = \Psi_m(I_0(\omega),\ldots,I_m(\omega)) = \Psi_m(-1,\ldots,-1,1,\ldots,1,-1)$, hence $\xi(\Omega_{l,m})$ is well defined for $1 \le l \le m \le T$.

Moreover I and therefore ξ are constant on each $\Omega_{l,T+1}$ such that $1 \le l \le T+1$, hence $\xi(\Omega_{l,T+1})$ is also well defined for each $1 \le l \le T+1$.

(ii) Since ξ is constant on each $\Omega_{\lambda,\nu}$, $(\lambda,\nu) \in \mathcal{I}$, which partition Ω, the $\mathfrak{F}_k$ conditional law of ξ is given, for all $0 \le k \le T$, by

$$\mathcal{L}_k(\xi) = \sum_{(\lambda,\nu)\in\mathcal{I}} \mathbb{Q}_k[\Omega_{\lambda,\nu}]\boldsymbol{\delta}_{\xi(\Omega_{\lambda,\nu})}.$$

By Lemma 3.6, $\mathbb{Q}_k[\Omega_{\lambda,\nu}]$ is constant on each $\Omega_{l,m}$, implying that $\mathcal{L}_k(\xi)$ is also constant on each $\Omega_{l,m}$, $(l,m) \in \mathcal{I}$. In particular, $\mathbb{E}_k\left[\xi\right], \mathbb{V}a\mathbb{R}_k(\xi)$ and $\mathbb{ES}_k(\xi)$ are constant on each $\Omega_{l,m}$. Last, we compute

$$\begin{aligned}\mathbb{E}_k[\xi] &= \sum_{(\lambda,\nu)\in\mathcal{I}} \xi(\Omega_{\lambda,\nu})\mathbb{Q}_k[\Omega_{\lambda,\nu}] \\ &= \sum_{(l,m)\in\mathcal{I}} \mathbb{1}_{\Omega_{l,m}} \sum_{(\lambda,\nu)\in\mathcal{I}} \xi(\Omega_{\lambda,\nu})\mathbb{Q}_k[\Omega_{\lambda,\nu}](\Omega_{l,m}).\end{aligned}$$

□

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ORCID

Cyril Bénézet https://orcid.org/0000-0003-1717-2012
Stéphane Crépey https://orcid.org/0000-0001-9409-1748
Dounia Essaket https://orcid.org/0009-0002-0734-8600

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